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THREE ANDERSON ACCELERATED ITERATIVE METHODS FOR SOLVING LARGE SCALE LINEAR EQUATIONS*

Xiaowei Jia
Empyrean Technology Co., Ltd, Beijing 100102, China
Email: jiaxw@empyrean.com.cn
Zikang Qin
Graduate School of China Academy Engineering Physics, Beijing 100083, China
Email: qinzikang23@gscaep.ac.cn
Hengbin An[1)]
National Key Laboratory of Computational Physics,
Institute of Applied Physics and Computational Mathematics, Beijing 100094, China;
CAEP Software Center for High Performance Numerical Simulation, Beijing 100088, China
Email: an_hengbin@iapcm.ac.cn

Abstract

Anderson acceleration is a kind of effective method for improving the convergence of the general fixed point iteration. In the linear case, Anderson acceleration can be used to improve the convergence rate of matrix splitting based iterative methods. In this paper, by using Anderson acceleration on general splitting iterative methods for linear systems, three classes of methods are given. The first one is obtained by directly applying Anderson acceleration on splitting iterative methods. For the second class of methods, Anderson acceleration is used periodically in the splitting iteration process. The third one is constructed by combining the Anderson acceleration and split iteration method in each iteration process. The key of this class of method is to determine a combination coefficient for Anderson acceleration and split iteration method. One optimal combination coefficient is given. Some theoretical results about the convergence of the considered three methods are established. Numerical experiments show that the proposed methods are effective.

Mathematics subject classification: 65F10, 65B05, 65H10.
Key words: Linear systems of equations, Split iteration, Fixed point iteration, Anderson acceleration, Iteration acceleration.

1. Introduction

In real applications of scientific and engineering computation, it is often needed to solve large sparse linear systems

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \tag{1.1}$$

where $\mathbb{R}$ is the set of all real numbers, and n is the problem size. The solution for large scale linear systems is often the dominant computational cost in many real applications. For solving the linear system (1.1), two classes of methods are often used. One is the direct methods, the other is the iterative methods.

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1) Corresponding author

Direct methods, which are typically based on the matrix decomposition techniques, including LU decomposition based methods, Cholesky decomposition based methods. This class of methods are suitable for small or medium size problem (for example, $n < 10^4$). The direct methods usually scale poorly with the increase of the problem size.

Iterative methods are often used to solve large scale problems. Generally the iterative methods contains at least three classes. The first class of iterative methods are the classical iterative methods, such as Richardson method, Jacobi method, Gauss-Seidel method, and successive over-relaxation (SOR) method. This class of methods are based on the explicit matrix splitting technique. Note that Hermitian and skew-Hermitian splitting (HSS) method [5] is also a matrix splitting based method. The second class of iterative methods are the Krylov methods [31], such as conjugate gradient (CG) method, generalized minimal residual (GMRES) method, biconjugate gradient stabilized (BiCGStab) method. The third class of iterative methods are the problem specific (usually PDE based) methods, such as multigrid (MG) method, domain decomposition method (DDM), and algebraic multigrid (AMG) method [37].

The Krylov methods are most popular for solving large scale sparse linear systems. The efficiency of the Krylov methods is strongly dependent on the preconditioning [7]. The classical iterative methods, including HSS method, can be used as solvers or as preconditioners. The MG/AMG and DDM methods are alternatives of effective preconditioners for solving PDE discretized systems. Because AMG can be used on general linear system, it is widely used as preconditioning method. However, AMG suffers from parallel scalability, because of complex operations such as coarsening, restriction and interpolation, etc. Furthermore, AMG is more sophisticated and difficult to be implemented and used than the classical iterative methods. Compared to AMG method, some classical iterative methods (for example, Jacobi method, block Jacobi method) are easier to be implemented and have good parallel scalability.

However, the convergence rate of the classical methods is typically quite slow. Therefore, in real applications, these kinds of methods are seldom used only to solve large scale problems, but are often combined with other methods. For example, Gauss-Seidel method and Jacobi method are often used as the smoothers in multi-grid method.

To improve the convergence rate of Jacobi method, two aspects of research work appeared recently. One aspect of work is the research about the relaxation technique for Jacobi iteration. Yang and Mittal [38] first proposed the scheduled relaxation Jacobi (SRJ) method, and their results show that SRJ can improve the convergence rate of Jacobi method 100 times. In SRJ method, relaxation parameters are set beforehand as a sequence and the relaxation sequence is applied periodically in the iteration. Later, Adsuara *et al.* [1] made some improvement for SRJ, and gave some further application tests for SRJ. Adsuara *et al.* [2] provided some theoretical analysis for SRJ, and showed that SRJ is equivalent to Richardson method in some sense. Maity and Singh [21] considered SRJ as preconditioners and showed the performance of the method on Poisson problems.

Another aspect of research work for improving the convergence rate of Jacobi iteration is to regard the method as a fixed point iteration process and apply Anderson acceleration (AA) method. Anderson acceleration method is a kind of accelerating technique to improve the general fixed point iteration. This algorithm is proposed by Anderson [4], and was studied in recent ten years by many researchers [10, 11, 26, 32–35, 39, 40]. Besides, Anderson acceleration has been used in many application areas [3, 13–15, 18, 19, 25, 36]. It is deserved to be pointed out that Walker and Ni [35] have discussed Anderson acceleration for linear case and they proved that AA is essentially equivalent to GMRES method in linear case. Besides, Potra and