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AN OSCILLATION-FREE DISCONTINUOUS GALERKIN METHOD FOR A NONLINEAR STOCHASTIC CONVECTION-DOMINATED DIFFUSION PROBLEM AND ITS ERROR ANALYSIS*

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Abstract

In this paper, an effective oscillation-free discontinuous Galerkin (DG) scheme for a nonlinear stochastic convection-dominated problem is formulated and analyzed. The proposed oscillation-free scheme is capable to capture the steep fronts of solution automatically and distinguish the influence of the convection domination and noise perturbation. Under proper regularity assumptions, the optimal convergence rates in space and time are rigorously proved with the techniques of variational solution and conditional expectation. In the numerical simulation, the classical SIPG scheme and the proposed oscillation-free DG scheme are both performed and compared. The numerical convergence rates tests are first carried out to verify the theoretical results. The benchmark tests having the steep behaviors are further provided to illustrate the effectiveness and robustness of our proposed oscillation-free DG scheme.

Mathematics subject classification: 60H15, 65M12, 65M15, 65M60.
Key words: Stochastic convection-domination, Discontinuous Galerkin method, Conditional expectation technique, Euler-Maruyama approach, Error estimates.

1. Introduction

The stochastic convection-dominated diffusion problems involving with the evolutionary characteristics of some specific physical quantity exist in many physical applications. However, due to the convection-dominated features and uncertainties, the steep behaviors and fluctuations of solutions may occur. In the deterministic case, many numerical works have already been studied to deal with the steep behaviors. Since the uncertainties are incorporated, it is imperative to develop a more robust numerical method to handle the convection-domination and noise perturbation to the system. In this paper, we develop an oscillation-free discontinuous Galerkin approach to solve the following Itô-type nonlinear stochastic convection-diffusion problem with homogeneous Dirichlet or periodic boundary condition. More precisely, we numerically solve for $u : [0,T] \times \mathcal{D} \times \Omega \to \mathbb{R}$ satisfying

$$\mathrm{d}u = \kappa \Delta u \,\mathrm{d}t - \nabla \cdot \boldsymbol{f}(u)\,\mathrm{d}t + \sigma(u)\,\mathrm{d}W \quad \text{in } (0,T] \times \mathcal{D} \times \Omega, \tag{1.1a}$$
$$u(0) = u_0 \quad \text{in } \mathcal{D} \times \Omega, \tag{1.1b}$$

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where $\mathcal{D}$ is a bounded convex polygonal domain in $\mathbb{R}^d$ $(d = 2, 3)$, κ a small diffusion coefficient and $T > 0$ a finite final time. The vector function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d$ is a convection flux. The operator σ satisfies the usual Lipschitz condition and the driving noise W is a trace class Q-Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ (see Section 2.1 for its precise definition). The initial condition $u_0 : \mathcal{D} \times \Omega \to \mathbb{R}$ is $\mathcal{F}_0$ measurable.

For the stochastic convection-dominated problem, there exist two main hard concerns, that is, the convection-domination and the noise perturbation which both affect the regularity of the system. As is well known, for deterministic convection-dominated models, their solutions with time evolution may exhibit steep moving fronts no matter how smooth their initial values are. The noise induced by the Q-Wiener process also results in the system having low regularity in time. In turn to the numerical methods, the convection-domination and noise perturbation both lead that most usual numerical methods become less robust or even unstable, i.e. spurious numerical oscillations. The design of a more robust numerical scheme for the stochastic convection-dominated problem becomes much more meaningful. Over several decades, researchers never stop developing the numerical techniques to control spurious oscillations encountered in the convection-dominated problem. As one of the successfully numerical methods, DG methods combining with strong stability preserving techniques [2, 20] and weighted essentially non-oscillatory schemes [41], are especially employed for capturing the discontinuous phenomena. Inspired by the local projection stabilization technique [9], the literature [31] introduced an effectively oscillation-free scheme with damping term based on the classical DG method for the scalar hyperbolic conservation laws. In this paper, we proposed a slightly modified version of the damping term introduced in [31] and applied it to the stochastic convection-dominated diffusion problem. Numerical analyses and simulations show that our proposed oscillation-free scheme is feasible and effective.

In the existing literature, the semigroup approach [14] and the variational method [32] are mainly used methods in the numerical and theoretical study on the stochastic partial differential equations. In the comparison between the two methods, the semigroup approach depends on the property of semigroup operator which has a higher requirement of regularity of the system. Since the stochastic convection-dominated diffusion equation may evolve into the steep behaviors, in this paper, the variational method is considered, which is more flexible and takes the advantages of utilizing weak formulation. Up to now, the efficiently numerical techniques for stochastic partial differential equations have been widely researched, concerning the spatial discretization methods including the finite difference methods [3, 8, 18], the spectral Galerkin approaches [7, 10, 12, 37], the finite element discretization [4, 13, 16, 19, 25, 26, 38, 39], the discontinuous Galerkin methods [11, 29, 35] and references therein, in which the Euler-Maruyama methods [17, 38] and Milstein schemes [21, 28], etc., are usually investigated for the temporal direction approximations. However, to the best of our knowledge, effective numerical methods for the stochastic convection-dominated problem are rarely studied.

The main contributions of our paper are summarized as:

(1) An effective and robust oscillation-free Scheme 2, which is composed of the oscillation-free DG method in space and the Euler-Maruyama discretization in time, is developed. The proposed numerical scheme can automatically detect the shock, track the discontinuity and control the numerical oscillation near the shock/discontinuity.

(2) In the numerical analysis, a conditional expectation technique is introduced to simplify the numerical analysis for the discrete system. With some regularity assumptions, the