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APPLICATION OF THE COMPLETE RADIATION BOUNDARY CONDITION AND THE RATIONAL ABSORBING BOUNDARY CONDITION IN $\mathbb{R}^2$ AND $\mathbb{R}^3$*

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Abstract

In this study, we explore two distinct rational approximations to the radiation condition for effectively solving time-harmonic wave propagation problems governed by the Helmholtz equation in $\mathbb{R}^d, d = 2$ or 3. First, we focus on the well-known complete radiation boundary condition (CRBC), which was developed for a transparent boundary condition for two-dimensional problems. The extension of CRBC to three-dimensional problems is a primary concern. Applications of CRBC require removing a near-cutoff region for a frequency range of a process to minimize reflection errors. To address the limitation faced by the CRBC application we introduce another absorbing boundary condition that avoids this demanding truncation. It is a new rational approximation to the radiation condition, which we call a rational absorbing boundary condition, that is capable of accommodating all types of propagating wave modes, including the grazing modes. This paper presents a comparative performance assessment of two approaches in two and three-dimensional spaces, providing insights into their effectiveness for practical application in wave propagation problems.

Mathematics subject classification: 65N30, 65N12, 74J20, 76Q05.
Key words: Absorbing boundary condition, Complete radiation condition, Rational absorbing boundary condition, Helmholtz equation.

1. Introduction

In this paper, we investigate absorbing boundary conditions (ABCs) based on two distinct rational approximations to the radiation condition for solving a time-harmonic wave propagation problem described by the Helmholtz equation

$$-\Delta u - k^2 u = f \quad \text{in } \mathbb{R}^d \tag{1.1}$$

with the radiation condition at infinity with $d = 2$ or 3, where k is a positive wavenumber and f is a compactly supported wave source. Finding accurate numerical solutions to the Helmholtz equation (1.1) is important in many areas of science and engineerings such as acoustics [44], electromagnetics [22], geophysics [40], and quantum mechanics [9].

For the numerical study for propagating solutions to the problem, it is necessary to truncate the infinite domain to a finite computation domain. This is achieved with the help of an absorbing boundary condition allowing propagating wave fields to pass though artificial boundaries resulting from the domain truncation as if no boundaries exist. There are many

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different types of approaches to develop the absorbing boundary conditions such as the perfectly matched layer (PML) method [5, 6, 27], Dirichlet-to-Neumann conditions [10, 20, 23, 28], approximate far-fields expansions [4,29], pole conditions [18,19,41], and ABCs based on rational approximations [8, 32].

For developing ABCs in the half space problem, where solutions propagate in the positive x-direction, we suppose that the support of f is in a half space for $x < -\delta$ with $\delta > 0$ and $(x, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Concerning an ABC on $x = 0$ that can capture the behavior of propagating wave fields along the positive x-direction, we can take the Fourier transform for the transverse variables $\tilde{x}$ of the Helmholtz equation (1.1)

$$\partial_x^2 \hat{u} + (k^2 - |\xi|^2)\hat{u} = 0,$$

where $\hat{u}(x, \xi)$ is the Fourier transform of $u(x, \tilde{x})$ with the Fourier variables ξ. Then a general solution can be written as

$$\hat{u} = Ae^{i\sqrt{k^2-|\xi|^2}x} + Be^{-i\sqrt{k^2-|\xi|^2}x} \tag{1.2}$$

with some constants A and B, where the square root is defined with a branch cut along the negative real axis. Specifically, the principal branch is chosen so that

$$-\frac{\pi}{2} < \arg\left(\sqrt{k^2 - |\xi|^2}\right) \leq \frac{\pi}{2}.$$

Under the time-harmonic assumption of the time variation, $e^{-i\omega t}$, at time t with angular frequency ω (so that the wavenumber k is determined by $k = \omega/c$ with the wave speed c), the solution radiating to the positive x-direction is required to have $B = 0$ and satisfy the Dirichlet-to-Neumann (DtN) boundary condition on $x = 0$,

$$\partial_x \hat{u} = i\sqrt{k^2 - |\xi|^2}\hat{u} := i\mu\hat{u} \quad \text{on } x = 0. \tag{1.3}$$

This DtN boundary condition is derived in the frequency domain, and the corresponding DtN boundary condition for the physical solution u can be recovered by using the inverse Fourier transformation, see [13] in more detail. The preferable ABC should be local and provide a good approximation to the square-root function. In this paper, we study two distinct ABCs approximating the square-root function in the DtN boundary condition by using rational functions in $\mathcal{R}_n$, the set of rational functions whose numerators and denominators are polynomials of degree at most n.

The first ABC we consider is the so-called complete radiation boundary condition, a well-established, transparent boundary condition in two-dimensional spaces as outlined in [12, 13, 24]. Our study primarily addresses the potential of implementing CRBC to three-dimensional wave propagation problems. In applying CRBC to open space problems, the method requires removing a near-cutoff region for a frequency range of $|\xi|$ on which reflection errors are to be minimized. To circumvent this issue, we propose the second ABC that eliminates the need for this demanding truncation. We develop a rational approximation to the radiation condition that can accommodate all propagating wave modes, including near-cutoff modes. The performance of both ABC approaches will be assessed and compared in two- and three-dimensional spaces.

The CRBC is inspired by Higdon's boundary condition [16,17], which is defined as a product of lowest order radiating conditions. Due to this product structure, Higdon's boundary condition is limited in its applicability as any high-order method. This limitation can be overcome by introducing auxiliary variables into Higdon's formula. In this way, the product operator