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Time Decay Estimates for Fourth-Order Schrödinger Operators in Dimension Three

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Abstract. This paper is concerned with the time decay estimates of the fourth order Schrödinger operator $H=\Delta^2+V(x)$ in dimension three, where $V(x)$ is a real valued decaying potential. Assume that zero is a regular point or the first kind resonance of H, and H has no positive eigenvalues, we established the following time optimal decay estimates of e^{-itH} with a regular term $H^{\alpha/4}$:

$$\|H^{\alpha/4}e^{-itH}P_{ac}(H)\|_{L^1-L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\le\alpha\le 3.$$

When zero is the second or third kind resonance of H, their decay will be significantly changed. We remark that such improved time decay estimates with the extra regular term $H^{\alpha/4}$ will be interesting in the well-posedness and scattering of nonlinear fourth order Schrödinger equations with potentials.

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1 Introduction

1.1 Backgrounds

In this paper, we will consider the time decay estimates of solution to the following fourth order Schrödinger equations in dimension three:

$$\begin{cases} iu_t=\Delta^2u+V(x)u, & (t,x)\in\mathbb{R}\times\mathbb{R}^3, \\ u(0,x)=u_0\in L^2(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $V(x)$ is a real valued potential on $\mathbb{R}^3$ satisfying $|V(x)|\lesssim(1+|x|)^{-\beta}$ for some $\beta>0$. It was well-known that the fourth order Schrödinger operator $H:=\Delta^2+V(x)$ is self-adjoint on $L^2(\mathbb{R}^3)$ by Kato-Rellich's theorem, and then the solution of Eq. (1.1) is given by $u(t)=e^{-itH}u_0$ by Stone's theorem.

As $V=0$, it follows that the free solution $u(t,x)=e^{-it\Delta^2}u_0$ can be expressed by Fourier transform :

$$e^{-it\Delta^2}u_0=\mathfrak{F}^{-1}(e^{-it|\xi|^4}\widehat{u_0})=\int_{\mathbb{R}^n}I_0(t,x-y)u_0(y)dy, \tag{1.2}$$

where $\hat{f}$ (or $\mathfrak{F}(f)$) denotes Fourier transform of f, $\mathfrak{F}^{-1}(f)$ denotes its inverse Fourier transform, and $I_0(t,x)=\mathfrak{F}^{-1}(e^{-it|\xi|^4})(x)$ is the kernel of $e^{-it\Delta^2}$. It was well-known that the kernel $I_0(t,x)$ satisfies the following optimal estimates for any $\alpha\in\mathbb{N}^n$ (see e.g., [2]):

$$|D^\alpha I_0(t,x)|\lesssim|t|^{-\frac{n+|\alpha|}{4}}\big(1+|t|^{-\frac14}|x|\big)^{-\frac{n-|\alpha|}{3}}, \quad |t|\neq0, \quad x\in\mathbb{R}^n, \tag{1.3}$$

where $D=(\partial_{x_1},\cdots,\partial_{x_n})$. Therefore by the (1.2) and Young's inequality, the (1.3) immediately implies that the following decay estimates hold:

$$\|D^\alpha e^{-it\Delta^2}\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)}\lesssim|t|^{-\frac{n+|\alpha|}{4}}, \quad |\alpha|\leq n. \tag{1.4}$$

Moreover, the regular term D^α of the inequality (1.4) can be replaced by fractional power $(-\Delta)^{\alpha/2}$ for any $0\leq\alpha\leq n$ (see e.g., [4]). It should be noticed that the decay estimates (1.4) with regular term D^α have played important roles in the