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Time Decay Estimates for Fourth-Order Schrödinger Operators in Dimension Three

Ping Li[1], Zijun Wan[2], Hua Wang[3] and Xiaohua Yao[4,*]

[1] *School of Informations and Mathematics, Yangtze University, Jingzhou, Hubei 434000, China*
[2] *School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, China*
[3] *Department of Mathematics and Hubei Province Key Laboratory of Mathematical Physics, Central China Normal University, Wuhan, Hubei 430079, China*
[4] *Department of Mathematics and Key Laboratory of Nonlinear Analysis and Applications (Ministry of Education), Central China Normal University, Wuhan, Hubei 430079, China*

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Abstract. This paper is concerned with the time decay estimates of the fourth order Schrödinger operator $H=\Delta^2+V(x)$ in dimension three, where $V(x)$ is a real valued decaying potential. Assume that zero is a regular point or the first kind resonance of H, and H has no positive eigenvalues, we established the following time optimal decay estimates of e^{-itH} with a regular term $H^{\alpha/4}$:

$$\|H^{\alpha/4}e^{-itH}P_{ac}(H)\|_{L^1-L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\leq\alpha\leq 3.$$

When zero is the second or third kind resonance of H, their decay will be significantly changed. We remark that such improved time decay estimates with the extra regular term $H^{\alpha/4}$ will be interesting in the well-posedness and scattering of nonlinear fourth order Schrödinger equations with potentials.

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*Corresponding author.
Emails: `liping@whu.edu.cn` (P. Li), `zijunwan@mails.ccnu.edu.cn` (Z. Wan), `wanghua_math@126.com` (H. Wang), `yaoxiaohua@ccnu.edu.cn` (X. Yao)

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1 Introduction

1.1 Backgrounds

In this paper, we will consider the time decay estimates of solution to the following fourth order Schrödinger equations in dimension three:

$$\begin{cases} iu_t=\Delta^2 u+V(x)u, & (t,x)\in\mathbb{R}\times\mathbb{R}^3,\\ u(0,x)=u_0\in L^2(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $V(x)$ is a real valued potential on $\mathbb{R}^3$ satisfying $|V(x)|\lesssim(1+|x|)^{-\beta}$ for some $\beta>0$. It was well-known that the fourth order Schrödinger operator $H:=\Delta^2+V(x)$ is self-adjoint on $L^2(\mathbb{R}^3)$ by Kato-Rellich's theorem, and then the solution of Eq. (1.1) is given by $u(t)=e^{-itH}u_0$ by Stone's theorem.

As $V=0$, it follows that the free solution $u(t,x)=e^{-it\Delta^2}u_0$ can be expressed by Fourier transform :

$$e^{-it\Delta^2}u_0=\mathfrak{F}^{-1}(e^{-it|\xi|^4}\widehat{u_0})=\int_{\mathbb{R}^n} I_0(t,x-y)u_0(y)dy, \tag{1.2}$$

where $\hat{f}$ (or $\mathfrak{F}(f)$) denotes Fourier transform of f, $\mathfrak{F}^{-1}(f)$ denotes its inverse Fourier transform, and $I_0(t,x)=\mathfrak{F}^{-1}(e^{-it|\xi|^4})(x)$ is the kernel of $e^{-it\Delta^2}$. It was well-known that the kernel $I_0(t,x)$ satisfies the following optimal estimates for any $\alpha\in\mathbb{N}^n$ (see e.g., [2]):

$$|D^\alpha I_0(t,x)|\lesssim|t|^{-\frac{n+|\alpha|}{4}}\big(1+|t|^{-\frac{1}{4}}|x|\big)^{-\frac{n-|\alpha|}{3}}, \quad |t|\neq 0, \quad x\in\mathbb{R}^n, \tag{1.3}$$

where $D=(\partial_{x_1},\cdots,\partial_{x_n})$. Therefore by the (1.2) and Young's inequality, the (1.3) immediately implies that the following decay estimates hold:

$$\|D^\alpha e^{-it\Delta^2}\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)}\lesssim|t|^{-\frac{n+|\alpha|}{4}}, \quad |\alpha|\le n. \tag{1.4}$$

Moreover, the regular term D^α of the inequality (1.4) can be replaced by fractional power $(-\Delta)^{\alpha/2}$ for any $0\le\alpha\le n$ (see e.g., [4]). It should be noticed that the decay estimates (1.4) with regular term D^α have played important roles in the well-posedness and scattering theory of the following the nonlinear fourth order Schrödinger equations:

$$\begin{cases} iu_t = \Delta^2 u + \lambda |u|^{p-1} u, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0 \in H^2(\mathbb{R}^n), \end{cases} \tag{1.5}$$

where $1<p<\infty$, $\lambda=+1$ denotes the defocusing equation, $\lambda=-1$ is the focusing one, see e.g., Pausder [21, 22], Miao, Xu and Zhao [19, 20] and references therein. Here inspired by (1.4), we will consider the decay estimates of e^{itH} with regular term $H^{\frac{\alpha}{4}}(0\le\alpha\le 3)$ in dimension three.

Recently, there exist several works devoted to the time decay estimates of the fourth order Schrödinger operator $H=\Delta^2+V(x)$. Feng et al. [10] firstly gave the asymptotic expansion of the resolvent $R_V=(H-z)^{-1}$ around zero threshold when zero is a regular point of H for $n\ge 5$ and $n=3$. Then they proved that Kato-Jensen type decay estimate of e^{-itH} is $(1+|t|)^{-n/4}$ for $n\ge 5$ and the $L^1\to L^\infty$ decay estimate is $\mathcal{O}(|t|^{-1/2})$ for $n=3$ in the regular case. Since then, Erdoğăn, Green and Toprak [8] for $n=3$ and Green-Toprak [11] for $n=4$ derived the asymptotic expansion of $R_V(z)$ near zero with the presence of resonance or eigenvalue, and established the L^1-L^∞ estimates for each kind of zero resonance. More recently, for dimension $n=1$, Soffer, Wu and Yao [24] proved the $L^1\to L^\infty$ estimate of e^{-itH} is $\mathcal{O}(|t|^{-\frac{1}{4}})$ whatever zero is a regular point or resonance. It's worth mentioning that different types of resonance don't change the optimal time decay rate of e^{-itH} in dimension one just at the cost of faster decay rate of the potential. Moreover, Li, Soffer and Yao [16] have obtained the L^1-L^∞ estimate of e^{-itH} in dimension two.

Furthermore, we also notice that there exist some interesting works on the L^p bounds of wave operators of the fourth order Schrödinger operator, see [13] for $n=3$ and [7] for $n>5$ in the regular case, also see [17] for $n=1$ in the zero regular or resonance cases.

1.2 Main results

We use the notation $a\pm:=a\pm\epsilon$ for some small but fixed $\epsilon>0$. For $a,b\in\mathbb{R}^+$, $a\lesssim b$ (or $a\gtrsim b$) means that there exists some constant $c>0$ such that $a\le cb$ (or $a\ge cb$). Moreover, if $a\lesssim b$ and $b\lesssim a$, then we write $a\sim b$.

Now we will present our main results on the decay estimates in the presence of zero resonance or eigenvalue. At first, we recall the definitions of zero resonances, also see Definition 3.1 below.

Let $\langle\cdot\rangle=(1+|\cdot|)^{1/2}$, and define the weighted L^2 spaces $L^2_s(\mathbb{R}^3)=\big\{f\colon \langle\cdot\rangle^s f\in L^2(\mathbb{R}^3)\big\}$.

We say that zero is *the first kind resonance of* H if there exists some nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ for some $\sigma>\frac{3}{2}$ but nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ for any $\sigma>\frac{1}{2}$ such that $H\phi=0$ in the distributional sense; zero is *the second kind resonance of* H if there exists some nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ for some $\sigma>\frac{1}{2}$ but no nonzero $\phi\in L^2(\mathbb{R}^3)$ such that $H\phi=0$; zero is *the third kind resonance of* H *(i.e., eigenvalue)* if there exists some nonzero $\phi\in L^2(\mathbb{R}^3)$ such that $H\phi=0$. We remark that such resonance solutions of $H\phi=0$ also can be characterized in the form of L^p spaces, see e.g., [8].

Now our main results are summarized in the theorem below.

Theorem 1.1. *Let* $|V(x)|\lesssim(1+|x|)^{-\beta}$ $(x\in\mathbb{R}^3)$ *with some* $\beta>0$. *Assume that* $H=\Delta^2+V(x)$ *has no positive embedded eigenvalues. Let* $P_{ac}(H)$ *denote the projection onto the absolutely continuous spectrum space of* H. *Then the following statements hold:*

(i). If zero is a regular point of H *and* $\beta>7$, *then*

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\le\alpha\le 3. \tag{1.6}$$

(ii). If zero is the first kind resonance of H *and* $\beta>11$, *then*

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\le\alpha\le 3. \tag{1.7}$$

(iii). If zero is the second kind resonance of H *and* $\beta>19$, *or the third kind resonance of* H *and* $\beta>23$, *then there is a time-dependent operator* $F_{\alpha,t}$ *satisfying*

$$\|F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha}{4}},\quad 0\le\alpha\le 3,$$

such that

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)-F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\le\alpha\le 3. \tag{1.8}$$

In particular, as $0\le\alpha\le 3$, *we have*

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty}\lesssim\begin{cases}|t|^{-\frac{3+\alpha}{4}}, & \text{if } |t|\le 1,\\ |t|^{-\frac{1+\alpha}{4}}, & \text{if } |t|>1.\end{cases} \tag{1.9}$$

Remark 1.1. Some comments on Theorem 1.1 are given as follows:

(i). When $V=0$, the estimate (1.6) can be reduced into the following decay estimates:

$$\big\||\nabla|^{\alpha}e^{-it\Delta^2}\big\|_{L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)}\lesssim|t|^{-\frac{3+|\alpha|}{4}},\quad 0\le\alpha\le 3, \tag{1.10}$$

which are actually sharp in view of the estimates (1.4). When $V\neq 0$ and $\alpha=0$, Theorem 1.1 are due to Erdoğan, Green and Toprak [8]. In order to deal with the cases with regular terms $H^{\alpha/4}$, The extra efforts are needed to establish the desired time decay bounds, we have used more detailed asymptotic expansions of the resolvent $R_V(\lambda^4)$ for λ near zero (see Theorem 3.1 below), and then used Littlewood-Paley method and oscillatory integral theory.

(ii). In additional, we also remark that time decay of $H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)$ in high energy part is always $|t|^{-\frac{3+\alpha}{4}}$ for $0\le\alpha\le 3$ for all resonances cases. Hence the obstructions to improved estimates are from lower energy part in the second and third and of resonance cases.

(iii). For the second kind of resonance case, we can get that

$$F_{\alpha,t}=K_{\alpha,t}+\mathcal{O}(|t|^{-\frac{2+\alpha}{4}}),$$

as $0\le\alpha\le 3$ (see (3.39) below), where $K_{\alpha,t}$ is a time-dependent operator defined in (3.33), which satisfies that

$$\|K_{\alpha,t}\|_{L^1\to L^\infty}\sim|t|^{-\frac{1+\alpha}{4}},\quad |t|>1.$$

It implies that the decay estimate (3.14) of $H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)$ is also optimal in general. Similarly, for the third kind of resonance we also have the analogous comments.

(iv). Recently, Goldberg and Green [13] have showed that the following wave operators

$$W_\pm=W_\pm(H,\Delta^2):=s-\lim_{t\to\pm\infty}e^{itH}e^{-it\Delta^2} \tag{1.11}$$

are bounded on $L^p(\mathbb{R}^3)$ for $1<p<\infty$ if zero is regular point of $H=\Delta^2+V$. However, due to the absence of the L^1 and L^∞ boundednesses of wave operators $W_\pm$ above in [13], also see [18] for the counterexamples of the endpoint cases. Therefore the time decay estimates in Theorem 1.1(i) can not be obtained by wave operator methods.

In Theorem 1.1, we assume that $H=\Delta^2+V$ has no any positive eigenvalues embedded into the absolutely continuous spectrum, which has been the indispensable condition in dispersive estimates. For Schrödinger operator $-\Delta+V$, Kato in [15] showed the absence of positive eigenvalues for $H=-\Delta+V$ with the decay potentials $V=o(|x|^{-1})$ as $|x|\to\infty$. For the four order Schrödinger operator $H=\Delta^2+V$, the situations seem to be subtle because there exist examples even with compactly supported smooth potentials such that the positive eigenvalues appear (see, e.g., [9]). On the other hand, more interestingly, a simple criterion has been proved in [9] that $H=\Delta^2+V$ has no any eigenvalues assume that the potential V is bounded and satisfies the repulsive condition (i.e., $(x\cdot\nabla)V\le 0$). Besides, we also notice that for a general selfadjoint operator $\mathcal{H}$ on $L^2(\mathbb{R}^n)$, even if $\mathcal{H}$ has a simple embedded eigenvalue, Costin and Soffer in [5] have proved that $\mathcal{H}+\epsilon V$ can kick off the eigenvalue located in a small interval under certain small perturbation of potential.

In order to obtain the decay estimates in Theorem 1.1, we will use the following Stone's formula for $0\leq\alpha\leq 3$,

$$H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)f(x)=\frac{2}{\pi i}\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}[R_V^+(\lambda^4)-R_V^-(\lambda^4)]f(x)d\lambda. \tag{1.12}$$

Note that the difference of the perturbed resolvents provides the spectral measure, hence we need to study carefully the resolvent operators $R_V(z)=(H-z)^{-1}$ by perturbations of the free resolvent $R_0(z)$ which have the following representation (see [10]):

$$R_0(z):=\big((-\Delta)^2-z\big)^{-1}=\frac{1}{2z^{\frac{1}{2}}}\big(R(-\Delta;z^{\frac{1}{2}})-R(-\Delta;-z^{\frac{1}{2}})\big),\quad z\in\mathbb{C}\setminus[0,\infty). \tag{1.13}$$

Here the resolvent

$$R(-\Delta;z^{\frac{1}{2}}):=(-\Delta-z^{\frac{1}{2}}).$$

For $\lambda\in\mathbb{R}^+$, we define the limiting resolvent operators by

$$R_0^\pm(\lambda):=R_0^\pm(\lambda\pm i0)=\lim_{\epsilon\to 0}\big(\Delta^2-(\lambda\pm i\epsilon)\big)^{-1}, \tag{1.14a}$$

$$R_V^\pm(\lambda):=R_V^\pm(\lambda\pm i0)=\lim_{\epsilon\to 0}\big(H-(\lambda\pm i\epsilon)\big)^{-1}. \tag{1.14b}$$

By using the representation (1.13) for $R_0(z)$ with $z=w^4$ for w in the first quadrant of the complex plane, and taking limits as $w\to\lambda$ and $w\to i\lambda$, we have

$$R_0^\pm(\lambda^4)=\frac{1}{2\lambda^2}\big(R^\pm(-\Delta;\lambda^2)-R(-\Delta;-\lambda^2)\big),\quad \lambda>0. \tag{1.15}$$

It is well-known that by the limiting absorption principle (see e.g., [1]), $R^\pm(-\Delta;\lambda^2)$ are well-defined as the bounded operators of $B(L^2_s,L^2_{-s})$ for any $s>1/2$, therefore $R_0^\pm(\lambda^4)$ are also well-defined between these weighted spaces. This property is extended to $R_V^\pm(\lambda^4)$ for $\lambda>0$ for certain decay bounded potentials, see [10].

The paper is organized as follows. In Section 2, we establish the dispersive bounds in the free case. In Section 3, we first recall the resolvent expansions when λ is near zero, then by Stone's formula, Littlewood-Paley method and oscillation integral we establish the low energy decay bounds of Theorem 1.1. In Section 4, we prove Theorem 1.1 in high energy.

2 The decay estimates for the free case

In this section, we are devote to establishing the decay bounds of the free case by Littlewood-Paley method and oscillatory integral theory.

Choosing a fixed even function $\varphi\in C_c^\infty(\mathbb{R})$ such that $\varphi(s)=1$ for $|s|\leq\frac{1}{2}$ and $\varphi(s)=0$ for $|s|\geq 1$. Let

$$\varphi_N(s)=\varphi(2^{-N}s)-\varphi(2^{-N+1}s),\quad N\in\mathbb{Z}.$$

Then $\varphi_N(s)=\varphi_0(2^{-N}s)$, $\mathrm{supp}\varphi_0\subset[\frac{1}{4},1]$ and

$$\sum_{N=-\infty}^{\infty}\varphi_0(2^{-N}s)=1,\quad s\in\mathbb{R}\setminus\{0\}. \tag{2.1}$$

By using Stone's formula, we have

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}e^{-it\Delta^2}f&=\frac{2}{\pi i}\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}[R_0^+(\lambda^4)-R_0^-(\lambda^4)]fd\lambda\\&=\frac{2}{\pi i}\sum_{N=-\infty}^{\infty}\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)[R_0^+(\lambda^4)-R_0^-(\lambda^4)]fd\lambda.\end{aligned}$$

Therefore, in order to obtain the L^1-L^∞ decay estimate of the $(-\Delta)^{\frac{\alpha}{2}}e^{-it\Delta^2}$, it suffices to estimate the following integral kernel for each N:

$$\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)R_0^\pm(\lambda^4)(x,y)d\lambda.$$

We first give a lemma which plays an important role in estimating our integrals mentioned in this paper. Since its proof is similar to Lemma 3.3 in [16], here we omit the details.

Lemma 2.1. *Let A be some subset of $\mathbb{Z}$. Suppose that $\Phi(s,z)$ is a function on $\mathbb{R}\times\mathbb{R}^m$ which is smooth for the first variable s, and satisfies for any $(s,z)\in[1/4,1]\times\mathbb{R}^m$,*

$$|\partial_s^k\Phi(2^Ns,z)|\lesssim 1,\quad k=0,1,\quad N\in A\in\mathbb{Z}.$$

Suppose that $\varphi_0(s)$ be a smoothing function of $\mathbb{R}$ defined in (2.1), $\Psi(z)$ is a nonnegative function on $\mathbb{R}^m$. Let

$$N_0=\left[\frac{1}{3}\log_2\frac{\Psi(z)}{|t|}\right]\quad \textit{for each}\quad z\in\mathbb{R}^m,\quad l\in\mathbb{R}\quad \textit{and}\quad t\neq 0,$$

we have

$$\Big|\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns\Psi(z)}s^l\varphi_0(s)\Phi(2^Ns,z)ds\Big|\lesssim\begin{cases}(1+|t|2^{4N})^{-\frac{1}{2}}, & \textit{if}\ \ |N-N_0|\leq 2,\\(1+|t|2^{4N})^{-1}, & \textit{if}\ \ |N-N_0|>2.\end{cases}$$

Throughout this paper, $\Theta_{N_0,N}(t)$ always denotes the following function:

$$\Theta_{N_0,N}(t):=\begin{cases}(1+|t|\cdot 2^{4N})^{-\frac{3}{2}}, & \text{if } \ |N-N_0|\leq 2,\\ (1+|t|\cdot 2^{4N})^{-2}, & \text{if } \ |N-N_0|>2,\end{cases} \tag{2.2}$$

where $N_0=\left[\frac{1}{3}\log_2\frac{\Psi(z)}{|t|}\right]$ and $\Psi(z)$ is a nonnegative real value function on $\mathbb{R}^m$.

Before we give the decay estimates of the operators $(-\Delta)^{\alpha/2}e^{it\Delta^2}$, we first recall that the free resolvent kernel of Laplacian $-\Delta$ in $\mathbb{R}^3$ (see [12]) is

$$R^{\pm}(-\Delta;\lambda^2)(x,y)=\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}, \tag{2.3}$$

hence by using the identity (1.15), it follows that

$$R_0^{\pm}(\lambda^4)(x,y)=\frac{1}{2\lambda^2}\Big(\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}-\frac{e^{-\lambda|x-y|}}{4\pi|x-y|}\Big). \tag{2.4}$$

Proposition 2.1. *Let $\Theta_{N_0,N}(t)$ be the function defined in* (2.2) *with $\Psi(z)=|x-y|$ and $z=(x,y)\in\mathbb{R}^6$. Then for each $x\neq y$ and $0\leq\alpha\leq 3$, we have*

$$\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)R_0^{\pm}(\lambda^4)(x,y)d\lambda\Big|\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t). \tag{2.5}$$

Moreover,

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}R_0^{\pm}(\lambda^4)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}}. \tag{2.6}$$

As a consequence, we obtain that

$$\|(-\Delta)^{\frac{\alpha}{2}}e^{-it\Delta^2}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}}. \tag{2.7}$$

Proof. We write

$$K_{0,N}^{\pm}(t,x,y):=\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)R_0^{\pm}(\lambda^4)(x,y)d\lambda.$$

Let $F^{\pm}(p)=\frac{e^{\pm ip}-e^{-p}}{p}$, $p\geq 0$, by the identity (2.4), we have

$$R_0^{\pm}(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^{\pm}(\lambda|x-y|). \tag{2.8}$$

Let $2^Ns=\lambda$, then

$$K_{0,N}^{\pm}(t,x,y)=\frac{2^{(3+\alpha)N}}{8\pi}\int_0^\infty e^{-it2^{4N}s^4}s^{2+\alpha}\varphi_0(s)F^{\pm}(2^Ns|x-y|)ds.$$

Note that $s\in \mathrm{supp}\varphi_0\subset[1/4,1]$, by using integration by parts, we have

$$\begin{aligned}
|K^{\pm}_{0,N}(t,x,y)|\lesssim&\frac{2^{(3+\alpha)N}}{1+|t|2^{4N}}\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\Big(s^{-1+\alpha}\varphi_0(s)F^{\pm}(2^Ns|x-y|)\Big)ds\Big|\\
\lesssim&\frac{2^{(3+\alpha)N}}{1+|t|2^{4N}}\bigg(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\big(s^{-1+\alpha}\varphi_0(s)\big)F^{\pm}(2^Ns|x-y|)ds\Big|\\
&+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-1+\alpha}\varphi_0(s)\partial_s\big(F^{\pm}(2^Ns|x-y|)\big)ds\Big|\bigg)\\
:=&\frac{2^{(3+\alpha)N}}{1+|t|2^{4N}}\Big(\big|\mathcal{E}^{\pm}_{01,N}(t,x,y)\big|+\big|\mathcal{E}^{\pm}_{02,N}(t,x,y)\big|\Big). \qquad (2.9)
\end{aligned}$$

For $\mathcal{E}^{\pm}_{02,N}(t,x,y)$. Let $r=|x-y|$, since

$$\partial_sF^{\pm}(2^Nsr)=s^{-1}2^Nsr(F^{\pm})'(2^Nsr):=e^{\pm i2^Nsr}s^{-1}F^{\pm}_1(2^Nsr),$$

where

$$F^{\pm}_1(p)=pe^{\mp ip}(F^{\pm})'(p)=\frac{(\pm ip-1)+(p+1)e^{-p\mp ip}}{p}.$$

Then we have

$$\mathcal{E}^{\pm}_{02,N}(t,x,y)=\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns|x-y|}s^{-2+\alpha}\varphi_0(s)F^{\pm}_1(2^Ns|x-y|)ds.$$

Note that

$$|\partial_s^kF^{\pm}_1(2^Ns|x-y|)|\lesssim 1,\quad k=0,1,$$

by Lemma 2.1 with $z=(x,y)$, $\Psi=|x-y|$ and $\Phi(2^Ns,z)=F^{\pm}_1(2^Ns|x-y|)$, we obtain that $\mathcal{E}^{\pm}_{02,N}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$.

Using similar processes, we obtain the same bounds for $\mathcal{E}^{\pm}_{01,N}$. Furthermore, by (2.9) we get that $K^{\pm}_{0,N}$ is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$. Thus we obtain that the estimate (2.7) holds.

Finally, in order to obtain (2.6), it's suffices to prove that for any $x\neq y$,

$$\sum_{N=-\infty}^{+\infty}|K^{\pm}_{0,N}(t,x,y)|\lesssim|t|^{-\frac{3+\alpha}{4}}. \qquad (2.10)$$

In fact, for $t\neq 0$, there exists $N_0'\in\mathbb{Z}$ such that $2^{N_0'}\sim|t|^{-\frac14}$. If $0\le\alpha<3$, then for any

$x \neq y$,

$$\begin{aligned}\sum_{N=-\infty}^{+\infty}|K^{\pm}_{0,N}(t,x,y)| &\lesssim \sum_{N=-\infty}^{+\infty} 2^{(3+\alpha)N}(1+|t|2^{4N})^{-\frac{3}{2}} \\ &\lesssim \sum_{N=-\infty}^{N_0'} 2^{(3+\alpha)N} + \sum_{N=N_0'+1}^{+\infty} 2^{(3+\alpha)N}(|t|2^{4N})^{-\frac{3}{2}} \lesssim |t|^{-\frac{3+\alpha}{4}}. \end{aligned} \tag{2.11}$$

If $\alpha=3$, then for any $x\neq y$,

$$\begin{aligned}\sum_{N=-\infty}^{+\infty}|K^{\pm}_{0,N}(t,x,y)| &\lesssim \sum_{|N-N_0|\leq 2} 2^{6N}(1+|t|2^{4N})^{-\frac{3}{2}} + \sum_{|N-N_0|>2} 2^{6N}(1+|t|2^{4N})^{-2} \\ &\lesssim |t|^{-\frac{3}{2}} + \sum_{N=-\infty}^{N_0'} 2^{6N} + \sum_{N=N_0'+1}^{+\infty} 2^{6N}(|t|2^{4N})^{-2} \lesssim |t|^{-\frac{3}{2}}. \end{aligned} \tag{2.12}$$

Hence, we obtain (2.10), which gives (2.6). Furthermore, the estimate (2.7) holds. This completes the proof. □

3 Low energy decay estimates

In this section, we are devote to establishing the decay estimates of $H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)$ for $0\leq\alpha\leq 3$ for low energy part. We first recall that the asymptotic expansions of the perturbed resolvent $R_V(\lambda^4)$ as λ near zero (see [8]), then by stone's formula we obtain the decay bounds of Theorem 1.1 for low energy part.

3.1 Asymptotic expansions of resolvent near zero

In this subsection, we recall that the asymptotic expansions of the perturbed resolvent $R_V(\lambda^4)$ when λ near zero, see [8] and also see [10] for the regular case.

By using the free resolvent kernel $R_0^{\pm}(\lambda^4)(x,y)$ in (2.4), we have the following expression:

$$\begin{aligned}R_0^{\pm}(\lambda^4)(x,y) =& \frac{a^{\pm}}{\lambda}I(x,y)+G_0(x,y)+a_1^{\pm}\lambda G_1(x,y)+a_3^{\pm}\lambda^3 G_3(x,y) \\ &+\lambda^4 G_4(x,y)+\sum_{k=5}^{N} a_k^{\pm}\lambda^k G_k(x,y)+\mathcal{O}\big(\lambda^{N+1}|x-y|^{N+2}\big), \end{aligned} \tag{3.1}$$

where

$$G_0(x,y)=-\frac{|x-y|}{8\pi},\quad G_1(x,y)=|x-y|^2,\quad G_3(x,y)=|x-y|^4, \tag{3.2a}$$

$$G_4(x,y)=-\frac{|x-y|^5}{4\pi\cdot 6!},\quad G_k(x,y)=|x-y|^{k+1},\quad k\geq 5, \tag{3.2b}$$

and the coefficients

$$a^{\pm}=\frac{1\pm i}{8\pi},\quad a_1^{\pm}=\frac{1\mp i}{8\pi\cdot 3!},\quad a_3^{\pm}=\frac{1\pm i}{8\pi\cdot 5!},\quad a_k^{\pm}=\frac{(-1)^{k+1}+(\pm i)^{k+2}}{8\pi\cdot(k+2)!},\quad (k\geq 5).$$

In the sequel, we also denote by G_k operators with the integral kernels $G_k(x,y)$ above. In particular, $G_0=(\Delta^2)^{-1}$.

Let $U(x)=\mathrm{sign}\big(V(x)\big)$ and $v(x)=|V(x)|^{1/2}$, then we have $V=Uv^2$ and the following symmetric resolvent identity

$$R_V^{\pm}(\lambda^4)=R_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v(M^{\pm}(\lambda))^{-1}vR_0^{\pm}(\lambda^4), \tag{3.3}$$

where

$$M^{\pm}(\lambda)=U+vR_0^{\pm}(\lambda^4)v.$$

Hence, we need to obtain the expansions for $(M^{\pm}(\lambda))^{-1}$.

Let $T=U+vG_0v$, and $P=\|V\|_{L^1}^{-1}v\langle v,\cdot\rangle$ denotes the orthogonal projection onto the span space by v. Now we introduce the type of resonances that may occur at the zero energy as follows.

Definition 3.1. *Let $Q=I-P$ and $T=U+vG_0v$.*

(i). If QTQ is invertible on QL^2, then we say that zero is a regular point of H. In this case, we define $D_0=(QTQ)^{-1}$ as an operator on QL^2.

(ii). Assume that QTQ is not invertible on QL^2. Let S_1 be the Riesz projection onto the kernel of QTQ. Then $QTQ+S_1$ is invertible on QL^2. In this case, we define $D_0=\big(QTQ+S_1\big)^{-1}$ as an operator on QL^2, which doesn't conflict with the previous definition since $S_1=0$ when zero is a regular point. We say that zero is the first kind resonance of H if

$$T_1:=S_1TPTS_1-\frac{\|V\|_{L^1}}{3\cdot(8\pi)^2}S_1vG_1vS_1 \tag{3.4}$$

is invertible on S_1L^2. We define $D_1=T_1^{-1}$ as an operator on S_1L^2.

(iii). Assume that T_1 is not invertible on S_1L^2. Let S_2 be the Riesz projection onto the kernel of T_1. Then T_1+S_2 is invertible on S_1L^2. In this case, we define $D_1=\big(T_1+S_2\big)^{-1}$ as an operator on S_1L^2, which doesn't conflict with previous definition since $S_2=0$ when zero is the first kind of resonance. We say that zero is the second kind resonance of H if

$$T_2:=S_2vG_3vS_2+\frac{10}{3\|V\|_{L^1}}S_2(vG_1v)^2S_2-\frac{10}{3\|V\|_{L^1}}S_2vG_1vTD_1TvG_1vS_2 \tag{3.5}$$

is invertible on S_2L^2. We define $D_2=T_2^{-1}$ as an operator on S_2L^2.

(iv). Finally if T_2 is not invertible on S_2L^2, we say that zero is the third kind resonance of H. In this case, the operator $T_3:=S_3vG_4vS_3$ is always invertible on S_3L^2 where S_3 be the Riesz projection onto the kernel of T_2, let $D_3=T_3^{-1}$ as an operator on S_3L^2. We define $D_2=(T_2+S_3)^{-1}$ as an operator on S_2L^2.

From the definition above, we have $S_1L^2\supseteq S_2L^2\supseteq S_3L^2$, which describe the zero energy resonance types of H as follows:

- zero is a regular point of H if and only if $S_1L^2=\{0\}$;
- zero is the first kind resonance of H if and only if $S_1L^2\neq\{0\}$ and $S_2L^2=\{0\}$;
- zero is the second kind resonance of H if and only if $S_2L^2\neq\{0\}$ and $S_3L^2=\{0\}$;
- zero is an eigenvalue of H (i.e., the third kind resonance) if and only if $S_3L^2\neq\{0\}$.

Note that these spectral subspaces S_jL^2 $(j=1,2,3)$ can be characterized by the distributional solution of $H\phi=0$ in [8], hence we also have the following statements:

- zero is the first kind resonance of H if there exists some nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ for $\sigma>\frac{3}{2}$ but no any nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ with $\sigma>\frac{1}{2}$ such that $H\phi=0$ in the distributional sense;
- zero is the second kind resonance of H if there exists some nonzero $\phi\in L^2_{-\sigma}(\mathbb{R}^3)$ for $\sigma>\frac{1}{2}$ but no any nonzero $\phi\in L^2$ such that $H\phi=0$ in the distributional sense;
- zero is the third kind resonance (i.e., eigenvalue) of H if there exists some nonzero $\phi\in L^2(\mathbb{R}^3)$ such that $H\phi=0$ in the distributional sense;
- zero is a regular point of H if zero is neither a resonance nor an eigenvalue of H.

In the following, we will give the specific characterizations of projection spaces S_jL^2 $(j=1,2,3)$ by the orthogonality of these projection operators S_j $(j=1,2,3)$, see also [8].

Lemma 3.1. *Let S_j $(j=1,2,3)$ be the projection operators given by Definition* 3.1. *Then*

(i). $f\in S_1L^2$ if and only if $f\in ker(QTQ)$. Moreover, $QTS_1=S_1TQ=0$.

(ii). $f\in S_2L^2$ if and only if

$$\begin{aligned} f\in ker(T_1)=&ker(S_1TPTS_1)\cap ker(S_1vG_1vS_1)\\ =&\{f\in S_1L^2\big|PTf=0,\ \langle x_iv,\ f\rangle=0,\ j=1,2,3\}. \end{aligned}$$

In particular, $TS_2=S_2T=0$, $QvG_1vS_2=S_2vG_1vQ=0$.

(iii). $f\in S_3L^2$ if and only if

$$f\in ker(T_2)=\{f\in S_2L^2\big|\langle x_ix_jv,\ f\rangle=0,\ i,j=1,2,3\}.$$

Furthermore, note that vG_0v is a Hilbert-Schmidt operator, and $T=U+vG_0v$ is the compact perturbation of U (see e.g., [6,11]). Hence S_1 is a finite-rank projection by the Fredholm alternative theorem. Notice that $S_3\leq S_2\leq S_1$, then all $S_j(j=1,2,3)$ are finite-rank operators. Hence the dimensions of these spaces $(S_1-S_2)L^2$, $(S_2-S_3)L^2$ and S_3L^2 corresponding to each zero resonance type, are finite. Moreover, by the definitions of S_j $(j=1,2,3)$, we have that $S_iD_j=D_jS_i=S_i$ $(i\geq j)$ and $S_iD_j=D_jS_i=D_j$ $(i<j)$.

Definition 3.2. *We say an operator $T:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ with kernel $T(\cdot,\cdot)$ is absolutely bounded if the operator with the kernel $|T(\cdot,\cdot)|$ is bounded from $L^2(\mathbb{R}^3)$ into itself.*

We remark that Hilbert-Schmidt and finite-rank operators are absolutely bounded operators. Moreover, we have the following proposition, see Lemma 4.3 in [8].

Proposition 3.1. *Let $|V(x)|\leq(1+|x|)^{-7-}$. Then QD_0Q is absolutely bounded.*

Now we will give asymptotic expansions of $\big(M^{\pm}(\lambda)\big)^{-1}$ as follows:

Theorem 3.1. *Let S_j $(j=1,2,3)$ be the operators defined in Definition* 3.1. *Assume that $|V(x)|\lesssim(1+|x|)^{-\beta}$ with some $\beta>0$. Then we have the following expansions of $\big(M^{\pm}(\lambda)\big)^{-1}$ in $L^2(\mathbb{R}^3)$ when $0<\lambda\ll1$.*

(i). If zero is a regular point of H and $\beta>7$, then

$$\big(M^{\pm}(\lambda)\big)^{-1}=QA^0_{0,1}Q+\Gamma_1(\lambda); \tag{3.6}$$

(ii). If zero is the first kind resonance of H and $\beta>11$, then

$$\big(M^{\pm}(\lambda)\big)^{-1}=\frac{S_1A^1_{-1,1}S_1}{\lambda}+\Big(S_1A^1_{0,1}+A^1_{0,2}S_1+QA^1_{0,3}Q\Big)+\Gamma_1(\lambda); \tag{3.7}$$

(iii). If zero is the second kind resonance of H and $\beta>19$, then

$$\begin{aligned}\big(M^{\pm}(\lambda)\big)^{-1}=&\frac{S_2A^2_{-3,1}S_2}{\lambda^3}+\frac{S_2A^2_{-2,1}S_1+S_1A^2_{-2,2}S_2}{\lambda^2}+\frac{S_2A^2_{-1,1}+A^2_{-1,2}S_2}{\lambda}\\&+\frac{S_1A^2_{-1,3}S_1}{\lambda}+\Big(S_1A^2_{0,1}+A^2_{0,2}S_1+QA^2_{0,3}Q\Big)+\Gamma_1(\lambda);\end{aligned} \tag{3.8}$$

(iv). If zero is the third kind resonance of H and $\beta>23$, then

$$\begin{aligned}\big(M^{\pm}(\lambda)\big)^{-1}=&\frac{S_3D_3S_3}{\lambda^4}+\frac{S_2A^3_{-3,1}S_2}{\lambda^3}+\frac{S_2A^3_{-2,1}S_1+S_1A^3_{-2,2}S_2}{\lambda^2}+\frac{S_2A^3_{-1,1}+A^3_{-1,2}S_2}{\lambda}\\&+\frac{S_1A^3_{-1,3}S_1}{\lambda}+\Big(S_1A^3_{0,1}+A^3_{0,2}S_1+QA^3_{0,3}Q\Big)+\Gamma_1(\lambda),\end{aligned} \tag{3.9}$$

where $A^k_{i,j}$ are λ-independent absolutely bounded operators in $L^2(\mathbb{R}^3)$; $\Gamma_1(\lambda)$ be a λ-dependent operator which may vary from line to line, and it satisfies

$$\big\|\Gamma_1(\lambda)\big\|_{L^2\to L^2}+\lambda\big\|\partial_\lambda\Gamma_1(\lambda)\big\|_{L^2\to L^2}+\lambda^2\big\|\partial^2_\lambda\Gamma_1(\lambda)\big\|_{L^2\to L^2}\lesssim\lambda.$$

The asymptotic expansions of $\big(M^{\pm}(\lambda)\big)^{-1}$ in $L^2(\mathbb{R}^3)$ above can be found in [8]. Here Theorem 3.1 take some different notations and expand more terms for our applications. see [3] for the details of proof.

3.2 Low energy decay estimates

In this subsection, we are devoting to establishing the low energy decay bounds for Theorem 1.1.

Below we use the smooth and even cut-off χ given by $\chi=1$ for $|\lambda|<\lambda_0\ll1$ and $\chi=0$ for $|\lambda|>2\lambda_0$, where λ_0 is some sufficiently small positive constant. In analysing the high energy later, we utilize the complementary cut-off $\widetilde{\chi}(\lambda):=1-\chi(\lambda)$.

Using the functional calculus and Stone's formula, one has

$$\begin{aligned} H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)f=&\frac{2}{\pi i}\int_0^{\infty} e^{-it\lambda^4}\lambda^{3+\alpha}[R_V^+(\lambda^4)-R_V^-(\lambda^4)]fd\lambda \\ =&\frac{2}{\pi i}\int_0^{\infty}\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}[R_V^+(\lambda^4)-R_V^-(\lambda^4)]fd\lambda \\ &+\frac{2}{\pi i}\int_0^{\infty}\widetilde{\chi}e^{-it\lambda^4}\lambda^{3+\alpha}(\lambda)[R_V^+(\lambda^4)-R_V^-(\lambda^4)]fd\lambda, \end{aligned} \tag{3.10}$$

where

$$\chi(\lambda)=\sum_{N=-\infty}^{N'}\varphi_0(2^{-N}\lambda) \quad \text{and} \quad \widetilde{\chi}(\lambda)=\sum_{N=N'+1}^{+\infty}\varphi_0(2^{-N}\lambda) \quad \text{for} \quad N'<0.$$

We remark that the choice of the constant N' depends on a sufficiently small neighbourhood of $\lambda=0$ in which the expansions of all resonance types in Theorem 3.1 hold.

Theorem 3.2 (Low Energy Dispersive Estimate). *Let $|V(x)|\lesssim(1+|x|)^{-\beta}$ ($x\in\mathbb{R}^3$) with some $\beta>0$. Assume that $H=(-\Delta)^2+V(x)$ has no positive embedded eigenvalues. Let $P_{ac}(H)$ denotes the projection onto the absolutely continuous spectrum space of H*

(i). If zero is a regular point and $\beta>7$, then

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}}, \quad 0\le\alpha\le3. \tag{3.11}$$

(ii). If zero is the first kind of resonance and $\beta>11$, then

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}}, \quad 0\le\alpha\le3. \tag{3.12}$$

(iii). If zero is the second kind of resonance and $\beta>19$, or the third kind of resonance and $\beta>23$, then there is a time-dependent operator $F_{\alpha,t}$ satisfying

$$\|F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha}{4}}, \quad 0\le\alpha\le3,$$

such that

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)-F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}}, \quad 0\le\alpha\le3. \tag{3.13}$$

In particular, for $0\le\alpha\le3$,

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)\|_{L^1\to L^\infty}\lesssim\begin{cases}|t|^{-\frac{3+\alpha}{4}}, & \text{if} \quad |t|\le1,\\ |t|^{-\frac{1+\alpha}{4}}, & \text{if} \quad |t|>1.\end{cases} \tag{3.14}$$

Before proving Theorem 3.2, we first give the following lemma, which has a crucial role in making use of cancellations of projection operators Q, S_j $(j=1,2,3)$ in the asymptotical expansions of resolvent $R_V(\lambda^4)$ as λ near zero, and will be used frequently to obtain the low energy dispersive estimate.

Lemma 3.2. *Assume that $x,y\in\mathbb{R}^3$ and $\lambda>0$. We define $w=w(x)=\frac{x}{|x|}$ for $x\neq 0$ and $w(x)=0$ for $x=0$. Let $\theta\in[0,1]$ and $|y|\cos\alpha=\langle y,w(x-\theta y)\rangle$, where $\alpha\equiv\alpha(x,y,\theta)$ is the angle between the vectors y and $x-\theta y$*

(i). If $F(p)\in C^1(\mathbb{R})$. Then

$$F(\lambda|x-y|)=F(\lambda|x|)-\lambda|y|\int_0^1\cos\alpha F'(\lambda|x-\theta y|)d\theta;$$

(ii). If $F(p)\in C^2(\mathbb{R})$ and $F'(0)=0$. Then

$$\begin{aligned}F(\lambda|x-y|)=&F(\lambda|x|)-\lambda\big\langle y,w(x)\big\rangle F'(\lambda|x|)\\&+\lambda^2|y|^2\int_0^1(1-\theta)\Big(\sin^2\alpha\frac{F'(\lambda|x-\theta y|)}{\lambda|x-\theta y|}+\cos^2\alpha F''(\lambda|x-\theta y|)\Big)d\theta;\end{aligned}$$

(iii). If $F(p)\in C^3(\mathbb{R})$ and $F'(0)=F''(0)=0$. Then

$$\begin{aligned}F(\lambda|x-y|)=&F(\lambda|x|)-\lambda\big\langle y,w(x)\big\rangle F'(\lambda|x|)+\frac{\lambda^2}{2}\Big[\Big(|y|^2-\big\langle y,w(x)\big\rangle^2\Big)\frac{F'(\lambda|x|)}{\lambda|x|}\\&+\big\langle y,w(x)\big\rangle^2F''(\lambda|x|)\Big]+\frac{\lambda^3|y|^3}{2}\int_0^1(1-\theta)^2\Big[3\cos\alpha\sin^2\alpha\Big(\frac{F'(\lambda|x-\theta y|)}{\lambda^2|x-\theta y|^2}\\&-\frac{F''(\lambda|x-\theta y|)}{\lambda|x-\theta y|}\Big)-\cos^3\alpha F^{(3)}(\lambda|x-\theta y|)\Big]d\theta.\end{aligned}$$

Proof. By the similar processes as the proof of Lemma 3.5 in [16], we can prove this lemma. Here we omit the details. $\square$

3.2.1 Regular case

In order to establish the lower energy estimate (3.11), recall that Stone's formula

$$\begin{aligned}H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)f=&\frac{2}{\pi i}\int_0^\infty e^{-it\lambda^4}\chi(\lambda)\lambda^{3+\alpha}[R_V^+(\lambda^4)-R_V^-(\lambda^4)]fd\lambda\\=&\sum_{N=-\infty}^{N_0'}\sum_{\pm}\frac{2}{\pi i}\int_0^\infty e^{-it\lambda^4}\varphi_0(2^{-N}\lambda)\lambda^{3+\alpha}R_V^\pm(\lambda^4)fd\lambda.\quad(3.15)\end{aligned}$$

If zero is a regular point of spectrum of H, using (3.3) and (3.6), we have

$$R_V^{\pm}(\lambda^4)=R_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(QA_{0,1}^0Q\Big)vR_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Gamma_1(\lambda)vR_0^{\pm}(\lambda^4). \tag{3.16}$$

Combining with Proposition 2.1, in order to obtain (3.11), it suffices to prove the following Propositions 3.2 and 3.3.

Proposition 3.2. *Assume that $|V(x)|\lesssim(1+|x|)^{-7-}$. Let $\Theta_{N_0,N}(t)$ be a function defined in* (2.2) *and $N<N'$. Then for each x,y and $0\le\alpha\le 3$, we have*

$$\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)v(QA_{0,1}^0Q)vR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t).$$

As a consequence,

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty \chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^{\pm}(\lambda^4)v(QA_{0,1}^0Q)vR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}}. \tag{3.17}$$

Proof. We write

$$K_{1,N}^{0,\pm}(t;x,y):=\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)vQA_{0,1}^1QvR_0^{\pm}(\lambda^4)\big](x,y)d\lambda.$$

Let

$$F^{\pm}(p)=\frac{e^{\pm ip}-e^{-p}}{p},\quad p\ge 0,$$

then

$$R_0^{\pm}(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^{\pm}(\lambda|x-y|).$$

By the orthogonality $Qv(x)=0$ and Lemma 3.2(i), we have

$$\begin{aligned}
&[R_0^{\pm}(\lambda^4)vQA_{0,1}^1QvR_0^{\pm}(\lambda^4)](x,y)\\
=&\frac{1}{64\pi^2\lambda^2}\int_{\mathbb{R}^6}F^{\pm}(\lambda|x-u_2|)[vQA_{0,1}^1Qv](u_2,u_1)F^{\pm}(\lambda|y-u_1|)du_1du_2\\
=&\frac{1}{64\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1\cos\alpha_2\cos\alpha_1(F^{\pm})'(\lambda|x-\theta_2u_2|)(F^{\pm})'(\lambda|y-\theta_1u_1|)d\theta_1d\theta_2\\
&\cdot|u_1||u_2|[vQA_{0,1}^1Qv](u_2,u_1)du_1du_2,
\end{aligned}$$

where $\cos\alpha_1=\cos\alpha(y,u_1,\theta_1)$ and $\cos\alpha_2=\cos\alpha(x,u_2,\theta_2)$. Furthermore, we have

$$\begin{aligned}
&K^{0,\pm}_{1,N}(t;x,y)\\
=&\frac{1}{64\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1\Big(\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)(F^\pm)'(\lambda|x-\theta_2u_2|)\\
&\cdot(F^\pm)'(\lambda|y-\theta_1u_1|)d\lambda\Big)\cos\alpha_2\cos\alpha_1 d\theta_1d\theta_2|u_1||u_2|[vQA^1_{0,1}Qv](u_2,u_1)du_1du_2\\
:=&\frac{1}{64\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1 E^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\cos\alpha_2\cos\alpha_1 d\theta_1d\theta_2\\
&\cdot|u_1||u_2|[vQA^1_{0,1}Qv](u_2,u_1)du_1du_2. \qquad (3.18)
\end{aligned}$$

Now we begin to estimate $E^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)$. In fact, let $s=2^{-N}\lambda$, then

$$\begin{aligned}
&E^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\\
=&2^{(4+\alpha)N}\int_0^\infty e^{-it2^{4N}s^4}s^{3+\alpha}\varphi_0(s)(F^\pm)'(2^Ns|x-\theta_2u_2|)(F^\pm)'(2^Ns|y-\theta_1u_1|)ds.
\end{aligned}$$

By using integration by parts, we have

$$\begin{aligned}
&|E^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|\\
\lesssim&\frac{2^{(4+\alpha)N}}{1+|t|2^{4N}}\Big(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\big(s^\alpha\varphi_0(s)\big)(F^\pm)'(2^Ns|x-\theta_2u_2|)(F^\pm)'(2^Ns|y-\theta_1u_1|)ds\Big|\\
&+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^\alpha\varphi_0(s)\partial_s\big((F^\pm)'(2^Ns|x-\theta_2u_2|)(F^\pm)'(2^Ns|y-\theta_1u_1|)\big)ds\Big|\Big)\\
:=&\frac{2^{(4+\alpha)N}}{1+|t|2^{4N}}\Big(|\mathcal{E}^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|+|\mathcal{E}^{0,\pm}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|\Big). \qquad (3.19)
\end{aligned}$$

We begin to compute $\mathcal{E}^{0,\pm}_{2,N}$. We have

$$\begin{aligned}
&|\mathcal{E}^{0,\pm}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|\\
\lesssim&\Big(\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-1+\alpha}\varphi_0(s)\cdot2^Ns|x-\theta_2u_2|(F^\pm)^{(2)}(2^Ns|x-\theta_2u_2|)(F^\pm)'(2^Ns|y-\theta_1u_1|)ds\Big|\\
&+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-1+\alpha}\varphi_0(s)\cdot2^Ns|y-\theta_1u_1|(F^\pm)'(2^Ns|x-\theta_2u_2|)(F^\pm)^{(2)}(2^Ns|y-\theta_1u_1|)ds\Big|\\
:=&|\mathcal{E}^{0,\pm}_{21,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|+|\mathcal{E}^{0,\pm}_{22,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|. \qquad (3.20)
\end{aligned}$$

For the first term $\mathcal{E}^{0,\pm}_{21,N}$. Since

$$\begin{aligned}
&2^Ns|x-\theta_2u_2|(F^\pm)^{(2)}(2^Ns|x-\theta_2u_2|)(F^\pm)'(2^Ns|y-\theta_1u_1|)\\
:=&e^{\pm i2^Ns|x-\theta_2u_2|}e^{\pm i2^Ns|y-\theta_1u_1|}F^\pm_2(2^Ns|x-\theta_2u_2|)F^\pm_1(2^Ns|y-\theta_1u_1|),
\end{aligned}$$

where

$$F_1^{\pm}(p)=e^{\mp ip}(F^{\pm})'(p)=\frac{(\pm ip-1)+(p+1)e^{-p\mp ip}}{p^2},$$
$$F_2^{\pm}(p)=pe^{\mp ip}(F^{\pm})^{(2)}(p)=\frac{(2\mp 2ip-p^2)-(2+2p+p^2)e^{-p\mp ip}}{p^2}.$$

Hence, we have

$$\begin{aligned}&\mathcal{E}^{0,\pm}_{21,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\\ =&\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x-\theta_2u_2|+|y-\theta_1u_1|)}s^{-1+\alpha}\varphi_0(s)F_2^{\pm}(2^Ns|x-\theta_2u_2|)F_1^{\pm}(2^Ns|y-\theta_1u_1|)ds.\end{aligned}$$

It is easy to check that

$$\Big|\partial_s^k\Big(F_2^{\pm}(2^Ns|x-\theta_2u_2|)F_1^{\pm}(2^Ns|y-\theta_1u_1|)\Big)\Big|\lesssim 1,\quad k=0,1.$$

By Lemma 2.1 with $z=(x,y,\theta_1,\theta_2,u_1,u_2)$ and

$$\Psi(z)=|x-\theta_2u_2|+|y-\theta_1u_1|,\quad \Phi(2^Ns;z)=F_2^{\pm}(2^Ns|x-\theta_2u_2|)F_1^{\pm}(2^Ns|y-\theta_1u_1|),$$

we obtain that $\mathcal{E}^{0,\pm}_{21,N}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. Similarly, we obtain that $\mathcal{E}^{0,\pm}_{22,N}$ is controlled by the same bound. Hence we get that $\mathcal{E}^{0,\pm}_{2,N}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$.

Similarly, we obtain that $\mathcal{E}^{0,\pm}_{1,N}$ is controlled by the same bound. By (3.19) we have

$$|E^{0,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|\lesssim 2^{(4+\alpha)N}\Theta_{N_0,N}(t). \tag{3.21}$$

Since $|V(x)|\lesssim(1+|x|)^{-7-}$, by using Hölder's inequality, (3.18) and (3.21), we obtain that

$$\begin{aligned}|K^{0,\pm}_{1,N}(t;x,y)|\lesssim&2^{(4+\alpha)N}\Big(\|u_1v(u_1)\|_{L^2}\|QA^0_{0,1}Q\|_{L^2\to L^2}\|u_2v(u_2)\|_{L^2}\Big)\Theta_{N_0,N}(t)\\ \lesssim&2^{(3+\alpha)N}\Theta_{N_0,N}(t).\end{aligned}$$

Finally, by the same summing way with the proof of (2.10), we get (3.17). $\square$

Proposition 3.3. *Assume that $|V(x)|\lesssim(1+|x|)^{-7-}$. Let $\Theta_{N_0,N}(t)$ be a function defined in (2.2) and $N<N'$. Then for each x,y and $0\le\alpha\le 3$,*

$$\begin{aligned}&\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)v\Gamma_1(\lambda)vR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\\ \lesssim&2^{(3+\alpha)N}\Theta_{N_0,N}(t).\end{aligned}\tag{3.22}$$

Moreover,

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^{\pm}(\lambda^4)v\Gamma_1(\lambda)vR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}}. \tag{3.23}$$

Proof. To get (3.22), it's equivalent to show that

$$\begin{aligned}&K^{0,\pm}_{2,N}(t;x,y)\\:=&\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big\langle[v\Gamma_1(\lambda)v]\big(R_0^\pm(\lambda^4)(*,y)\big)(\cdot),\ (R_0^\pm)^*(\lambda^4)(x,\cdot)\big\rangle d\lambda\end{aligned}$$

is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$.

Let $F^\pm(p)=\frac{e^{\pm ip}-e^{-p}}{p}$, $p\geq 0$, Then

$$R_0^\pm(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^\pm(\lambda|x-y|).$$

Thus we have

$$\begin{aligned}&\big\langle[v\Gamma_1(\lambda)v]\big(R_0^\pm(\lambda^4)(*,y)\big)(\cdot),\ R_0^\mp(\lambda^4)(x,\cdot)\big\rangle\\=&\frac{1}{64\pi^2\lambda^2}\big\langle[v\Gamma_1(\lambda)v]\big(F^\pm(\lambda|*-y|)\big)(\cdot),\ F^\mp(\lambda|x-\cdot|)\big\rangle\\:=&\frac{1}{64\pi^2\lambda^2}E_2^{0,\pm}(\lambda;x,y).\end{aligned}$$

Let $2^Ns=\lambda$, then

$$K^{0,\pm}_{2,N}(t;x,y)=\frac{2^{(2+\alpha)N}}{64\pi^2}\int_0^\infty e^{-it2^{4N}s^4}s^{1+\alpha}\varphi_0(s)E_2^{0,\pm}(2^Ns;x,y)ds.$$

Since $s\in\operatorname{supp}\varphi_0\subset[\frac14,1]$, by using integration by parts we have

$$\begin{aligned}|K^{0,\pm}_{2,N}(t;x,y)|\lesssim&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\bigg(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\big(s^{-2+\alpha}\varphi_0(s)\big)E_2^{0,\pm}(2^Ns;x,y)ds\Big|\\&+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\partial_s\big(E_2^{0,\pm}(2^Ns;x,y)\big)ds\Big|\bigg)\\:=&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\Big(|\mathcal{E}^{0,\pm}_{1,N}(t;x,y)|+|\mathcal{E}^{0,\pm}_{2,N}(t;x,y)|\Big).\end{aligned}\tag{3.24}$$

We first estimate $\mathcal{E}^{0,\pm}_{2,N}$. Note that

$$\begin{aligned}\partial_s\big(E_2^{0,\pm}(2^Ns;x,y)\big)=&\big\langle[v\partial_s\Gamma_1(2^Ns)v]\big(F^\pm(2^Ns|*-y|)\big)(\cdot),\ F^\mp(2^Ns|x-\cdot|)\big\rangle\\&+\big\langle[v\Gamma_1(2^Ns)v]\big(\partial_sF^\pm(2^Ns|*-y|)\big)(\cdot),\ F^\mp(2^Ns|x-\cdot|)\big\rangle\\&+\big\langle[v\Gamma_1(2^Ns)v]\big(F^\pm(2^Ns|*-y|)\big)(\cdot),\ \partial_sF^\mp(2^Ns|x-\cdot|)\big\rangle\\:=&E_{21}^{0,\pm}(2^Ns;x,y)+E_{22}^{0,\pm}(2^Ns;x,y)+E_{23}^{0,\pm}(2^Ns;x,y).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{E}^{0,\pm}_{2,N}(t;x,y)=&\int_0^\infty e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\Big(E^{0,\pm}_{21}(2^Ns;x,y)\\&+E^{0,\pm}_{22}(2^Ns;x,y)+E^{0,\pm}_{23}(2^Ns;x,y)\Big)ds\\:=&\mathcal{E}^{0,\pm}_{21,N}(t;x,y)+\mathcal{E}^{0,\pm}_{22,N}(t;x,y)+\mathcal{E}^{0,\pm}_{23,N}(t;x,y).\end{aligned}$$

At first, we deal with the first term $\mathcal{E}^{0,\pm}_{21,N}(t;x,y)$. Since

$$\lambda\|\partial_\lambda\Gamma_1(\lambda)\|_{L^2\to L^2}+\lambda^2\|\partial^2_\lambda\Gamma_1(\lambda)\|_{L^2\to L^2}\lesssim\lambda,$$

then

$$\big\|\partial^k_s\big(\partial_s\Gamma_1(2^Ns)\big)\big\|_{L^2\to L^2}\lesssim 2^N,\quad k=0,1.$$

We can check that

$$\big|\partial^k_s\big(E^{0,\pm}_{21}(2^Ns;x,y)\big)\big|\lesssim 2^N,\quad k=0,1.$$

By integration by parts, we obtain that $\mathcal{E}^{0,\pm}_{21,N}(t;x,y)$ is bounded by $2^N(1+|t|2^{4N})^{-1}$.

Next we deal with $\mathcal{E}^{0,\pm}_{22,N}(t;x,y)$. Let

$$F^{\mp}_0(p)=\frac{1-e^{-p}e^{\pm ip}}{p},$$

then $F^{\mp}(p)=e^{\mp ip}F^{\mp}_0(p)$. Since

$$\partial_sF^{\pm}(2^Ns|*-y|)=2^N|*-y|(F^{\pm})'(2^Ns|*-y|):=e^{\pm i2^Ns|*-y|}s^{-1}F^{\pm}_1(2^Ns|*-y|),$$

where

$$F^{\pm}_1(p)=pe^{\mp ip}(F^{\pm})'(p)=\frac{(\pm ip-1)+(p+1)e^{-p}e^{\mp ip}}{p}.$$

Moreover,

$$\begin{aligned}E^{0,\pm}_{22}(2^Ns;x,y)=&e^{\pm i2^Ns(|x|+|y|)}s^{-1}\Big\langle[v\Gamma_1(2^Ns)v]\big(e^{\pm i2^Ns(|*-y|-|y|)}F^{\pm}_1(2^Ns|*-y|)\big)(\cdot),\\&e^{\mp i2^Ns(|x-\cdot|-|x|)}F^{\mp}_0(2^Ns|x-\cdot|)\Big\rangle\\:=&e^{\pm i2^Ns(|x|+|y|)}s^{-1}\widetilde{E}^{0,\pm}_{22}(2^Ns;x,y).\end{aligned}$$

Hence, we have

$$\mathcal{E}^{0,\pm}_{22,N}(t;x,y)=\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x|+|y|)}s^{-3+\alpha}\varphi_0(s)\big(2^{-N}\widetilde{E}^{0,\pm}_{22}(2^Ns;x,y)\big)ds.$$

Note that

$$\begin{aligned}&\left|\partial_s^k\left(e^{\pm i2^N s(|*-y|-|y|)}F_1^{\pm}(2^N s|*-y|)\right)\right|\lesssim 2^{kN}\langle *\rangle\lesssim\langle *\rangle, &&k=0,1,\\&\left|\partial_s^k\left(e^{\mp i2^N s(|x-\cdot|-|x|)}F_0^{\mp}(2^N s|x-\cdot|)\right)\right|\lesssim 2^{kN}\langle \cdot\rangle\lesssim\langle \cdot\rangle, &&k=0,1.\end{aligned}$$

Since $|V(x)|\lesssim(1+|x|)^{-7-}$, by Hölder's inequality, we have

$$\left|\partial_s^k\widetilde{E}_{22}^{0,\pm}(2^N s;x,y)\right|\lesssim\sum_{k=0}^{1}\left\|v(\cdot)\langle\cdot\rangle^{1-k}\right\|_{L^2}^2\left\|\partial_s^k\Gamma_1(2^N s)\right\|_{L^2\to L^2}\lesssim 2^N.$$

By Lemma 2.1 with

$$z=(x,y),\quad \Psi(z)=|x|+|y|\quad\text{and}\quad \Phi(2^N s;z)=\widetilde{E}_{22}^{0,\pm}(2^N s;x,y),$$

we obtain that $\mathcal{E}_{22,N}^{0,\pm}$ is bounded by $2^N(1+|t|2^{4N})\Theta_{N_0,N}(t)$. Similar to get that $\mathcal{E}_{23,N}^{0,\pm}$ is controlled by the same bound. Hence we obtain that $\mathcal{E}_{2,N}^{0,\pm}$ is bounded by $2^N(1+|t|2^{4N})\Theta_{N_0,N}(t)$.

Similarly, we obtain that $\mathcal{E}_{1,N}^{0,\pm}$ is controlled by the same bounds. By (3.24), we immediately obtain that $K_{2,N}^{0,\pm}$ is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$. Hence we obtain that (3.22) holds.

Finally, by the same argument with the proof of (2.10), we immediately get (3.23). □

3.2.2 The first kind of resonance

If zero is the first kind of resonance of H, by using (3.3) and (3.7) one has

$$\begin{aligned}R_V^{\pm}(\lambda^4)=&R_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(\lambda^{-1}S_1A_{-1,1}^1S_1\Big)vR_0^{\pm}(\lambda^4)\\&-R_0^{\pm}(\lambda^4)v\Big(S_1A_{0,1}^1+A_{0,2}^1S_1+QA_{0,3}^1Q\Big)vR_0^{\pm}(\lambda^4)\\&-R_0^{\pm}(\lambda^4)v\Gamma_1(\lambda)vR_0^{\pm}(\lambda^4).\end{aligned}\tag{3.25}$$

In order to obtain the estimates (3.12), compared with the analysis of regular case, it is enough to estimate to prove the following proposition.

Proposition 3.4. *Assume that $|V(x|)\lesssim(1+|x|)^{-11-}$. Then for $0\le\alpha\le 3$,*

$$\begin{aligned}&\sup_{x,y\in\mathbb{R}^3}\left|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\lambda^{-1}\left[R_0^{\pm}(\lambda^4)vS_1A_{-1,1}^1S_1vR_0^{\pm}(\lambda^4)\right](x,y)d\lambda\right|\\&\lesssim|t|^{-\frac{3+\alpha}{4}},\\&\sup_{x,y\in\mathbb{R}^3}\left|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\left[R_0^{\pm}(\lambda^4)vS_1A_{0,1}^1vR_0^{\pm}(\lambda^4)\right](x,y)d\lambda\right|\\&\lesssim|t|^{-\frac{3+\alpha}{4}}.\end{aligned}$$

Proof. Here we use the orthogonality $S_1v=0$, by the same argument with the proof of Proposition 3.2, we obtain that this proposition holds. $\square$

3.2.3 The second kind of resonance

If zero is the second kind of resonance of H, using (3.3) and (3.8) one has

$$\begin{aligned}R_V^{\pm}(\lambda^4)=&R_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(\lambda^{-3}S_2A^2_{-3,1}S_2\Big)vR_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(\lambda^{-2}S_2A^2_{-2,1}S_1\\&+\lambda^{-2}S_1A^2_{-2,2}S_2\Big)vR_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(\lambda^{-1}S_2A^2_{-1,1}+\lambda^{-1}A^2_{-1,2}S_2\\&+\lambda^{-1}S_1A^2_{-1,3}S_1\Big)vR_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)v\Big(S_1A^2_{0,1}+A^2_{0,2}S_1+QA^2_{0,3}Q\Big)vR_0^{\pm}(\lambda^4)\\&-R_0^{\pm}(\lambda^4)v\Gamma_1(\lambda)vR_0^{\pm}(\lambda^4).\end{aligned}\tag{3.26}$$

In order to prove Theorem 3.2(iii), combining with the proof of regular case and the first kind resonance, it is enough to estimate integrals with the following three terms:

$$\Omega_{2,1}(\lambda):=R_0^{\pm}(\lambda^4)v\big(\lambda^{-3}S_2A^2_{-3,1}S_2\big)vR_0^{\pm}(\lambda^4),\tag{3.27a}$$
$$\Omega_{2,2}(\lambda):=R_0^{\pm}(\lambda^4)v\big(\lambda^{-2}S_2A^2_{-2,1}S_1\big)vR_0^{\pm}(\lambda^4),\tag{3.27b}$$
$$\Omega_{2,3}(\lambda):=R_0^{\pm}(\lambda^4)v\big(\lambda^{-1}S_2A^2_{-1,1}\big)vR_0^{\pm}(\lambda^4).\tag{3.27c}$$

Since $(F^{\pm})'(0)\neq 0$, where

$$R_0^{\pm}(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^{\pm}(\lambda|x-y|),$$

so it doesn't satisfy the condition of Lemma 3.2(ii), hence we can't make full use of the orthogonality of S_2 (i.e., $S_2x_iv=0$, $i=1,2,3$). In order to using orthogonality $S_2x_jv=0$ to improve the time decay of $H^{\alpha/4}e^{-itH}P_{ac}(H)$, we will subtract some specific operator to satisfy the conditions of Lemma 3.2(ii). Then we can make full use of the orthogonality of S_2.

Let

$$\widetilde{F}^{\pm}(p)=\frac{e^{\pm ip}-e^{-p}}{p}+p,\quad p\in\mathbb{R}.$$

Recall that $G_0=\frac{|x-y|}{8\pi}$, then

$$R_0^{\pm}(\lambda^4)(x,y)-G_0=\frac{1}{8\pi\lambda}\widetilde{F}^{\pm}(\lambda|x-y|),\tag{3.28}$$

$\widetilde{F}^{\pm}(p)\in C^2(\mathbb{R})$ and $(\widetilde{F}^{\pm})'(0)=0$. Hence $\widetilde{F}^{\pm}(p)$ satisfies the condition of Lemma 3.2(ii). We now begin to estimates the terms $\Omega_{2,i}(\lambda)$ $(i=1,2,3)$ in (3.27).

Firstly, we deal with the first term $\Omega_{2,1}(\lambda)$ in (3.27). We have

$$\begin{aligned}\Omega_{2,1}(\lambda)=&\big(R_0^{\pm}(\lambda^4)-G_0\big)v(\lambda^{-3}S_2A^2_{-3,1}S_2)v\big(R_0^{\pm}(\lambda^4)-G_0\big)\\&+\big(R_0^{\pm}(\lambda^4)-G_0\big)v(\lambda^{-3}S_2A^2_{-3,1}S_2)\times vG_0\\&+G_0v(\lambda^{-3}S_2A^2_{-3,1}S_2)v\big(R_0^{\pm}(\lambda^4)-G_0\big)+G_0v(\lambda^{-3}S_2A^2_{-3,1}S_2)vG_0\\:=&\Gamma^2_{-3,1}(\lambda)+\Gamma^2_{-3,2}(\lambda)+\Gamma^2_{-3,3}(\lambda)+\Gamma^2_{-3,4}(\lambda).\end{aligned}\tag{3.29}$$

Proposition 3.5. *Assume* $|V(x)|\lesssim(1+|x|)^{-19-}$. *Let* $\Gamma^2_{-3,j}(\lambda)$ $(j=1,2,3)$ *be operators defined in* (3.29). *Then*

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-3,1}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\le\alpha\le3,\tag{3.30a}$$

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-3,j}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+\alpha}{4}},\quad 0\le\alpha\le4,\quad j=2,3.\tag{3.30b}$$

Proof. (i) We first show the first integral estimates. We write

$$\begin{aligned}&\widetilde{K}^{2,\pm}_{1,N}(t;x,y)\\=&\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[\big(R_0^{\pm}(\lambda^4)-G_0\big)v(\lambda^{-3}S_2A^2_{-3,1}S_2)v\big(R_0^{\pm}(\lambda^4)-G_0\big)\big](x,y)d\lambda.\end{aligned}$$

Recall that

$$R_0^{\pm}(\lambda^4)(x,y)-G_0=\frac{1}{8\pi\lambda}\widetilde{F}^{\pm}(\lambda|x-y|)$$

in (3.28), then by Lemma 3.2(ii) and the orthogonality $S_2x_jv(x)=S_2v(x)=0$ $(j=1,2,3)$, we have

$$\begin{aligned}&\big[\big(R_0^{\pm}(\lambda^4)-G_0\big)v(\lambda^{-3}S_2A^2_{-3,1}S_2)v\big(R_0^{\pm}(\lambda^4)-G_0\big)\big](x,y)\\=&\frac{1}{64\pi^2\lambda^5}\int_{\mathbb{R}^6}\widetilde{F}^{\pm}(\lambda|x-u_2|)v(u_2)(S_2A^2_{-3,1}S_2)(u_2,u_1)v(u_1)\widetilde{F}^{\pm}(\lambda|y-u_1|)du_1du_2\\=&\frac{1}{64\pi^2\lambda}\int_{\mathbb{R}^6}\int_0^1\int_0^1(1-\theta_1)(1-\theta_2)\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|x-\theta_2u_2|)}{\lambda|x-\theta_2u_2|}\sin^2\alpha_2+(\widetilde{F}^{\pm})^{(2)}(\lambda|x-\theta_2u_2|)\cos^2\alpha_2\Big)\\&\times\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|y-\theta_1u_1|)}{\lambda|y-\theta_1u_1|}\sin^2\alpha_1+(\widetilde{F}^{\pm})^{(2)}(\lambda|y-\theta_1u_1|)\cos^2\alpha_1\Big)d\theta_1d\theta_2|u_1|^2|u_2|^2v(u_2)v(u_1)\\&\times(S_2A^2_{-3,1}S_2)(u_2,u_1)du_1du_2.\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\widetilde{K}^{2,\pm}_{1,N}(t;x,y)\\
=&\frac{1}{64\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1\Big[\int_0^\infty e^{-it\lambda^4}\lambda^{2+\alpha}\varphi_0(2^{-N}\lambda)\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|x-\theta_2u_2|)}{\lambda|x-\theta_2u_2|}\sin^2\alpha_2\\
&+(\widetilde{F}^{\pm})^{(2)}(\lambda|x-\theta_2u_2|)\cos^2\alpha_2\Big)\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|y-\theta_1u_1|)}{\lambda|y-\theta_1u_1|}\sin^2\alpha_1+(\widetilde{F}^{\pm})^{(2)}(\lambda|y-\theta_1u_1|)\\
&\times\cos^2\alpha_1\Big)d\lambda\Big](1-\theta_1)(1-\theta_2)|u_1|^2|u_2|^2v(u_2)v(u_1)(S_2A^2_{-3,1}S_2)(u_2,u_1)d\theta_1d\theta_2du_1du_2.
\end{aligned}$$

Let

$$\begin{aligned}
&\widetilde{E}^{2,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\\
=&\int_0^\infty e^{-it\lambda^4}\lambda^{2+\alpha}\varphi_0(2^{-N}\lambda)\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|x-\theta_2u_2|)}{\lambda|x-\theta_2u_2|}\sin^2\alpha_2+(\widetilde{F}^{\pm})^{(2)}(\lambda|x-\theta_2u_2|)\cos^2\alpha_2\Big)\\
&\quad\times\Big(\frac{(\widetilde{F}^{\pm})'(\lambda|y-\theta_1u_1|)}{\lambda|y-\theta_1u_1|}\sin^2\alpha_1+(\widetilde{F}^{\pm})^{(2)}(\lambda|y-\theta_1u_1|)\cos^2\alpha_1\Big)d\lambda,
\end{aligned}$$

then

$$\begin{aligned}
\big|\widetilde{K}^{2,\pm}_{1,N}(t;x,y)\big|\lesssim&\int_{\mathbb{R}^6}\int_0^1\int_0^1\big|\widetilde{E}^{2,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\big||u_1|^2|u_2|^2|v(u_2)v(u_1)|\\
&\times|(S_2A^2_{-3,1}S_2)(u_2,u_1)|d\theta_1d\theta_2du_1du_2. \qquad (3.31)
\end{aligned}$$

Let

$$s=2^{-N}\lambda,\quad r_1=2^N|y-\theta_1u_1|\quad \text{and}\quad r_2=2^N|x-\theta_2u_2|,$$

then

$$\begin{aligned}
&\widetilde{E}^{2,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\\
=&2^{(3+\alpha)N}\int_0^\infty e^{-it2^{4N}s^4}s^{2+\alpha}\varphi_0(s)\Big(\frac{(\widetilde{F}^{\pm})'(2^Ns|x-\theta_2u_2|)}{2^Ns|x-\theta_2u_2|}\sin^2\alpha_2+(\widetilde{F}^{\pm})^{(2)}(2^Ns|x-\theta_2u_2|)\\
&\times\cos^2\alpha_2\Big)\Big(\frac{(\widetilde{F}^{\pm})'(2^Ns|y-\theta_1u_1|)}{2^Ns|y-\theta_1u_1|}\sin^2\alpha_1+(\widetilde{F}^{\pm})^{(2)}(2^Ns|y-\theta_1u_1|)\cos^2\alpha_1\Big)ds\\
=&2^{(3+\alpha)N}\int_0^\infty e^{-it2^{4N}s^4}s^{2+\alpha}\varphi_0(s)\prod_{j=1}^2\Big(\frac{(\widetilde{F}^{\pm})'(r_js)}{r_js}\sin^2\alpha_j+(\widetilde{F}^{\pm})^{(2)}(r_js)\cos^2\alpha_j\Big)ds.
\end{aligned}$$

By integration by parts, we obtain that

$$
\begin{aligned}
&|\widetilde{E}^{2,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)| \\
\lesssim&\frac{2^{(3+\alpha)N}}{1+|t|2^{4N}}\Big(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\Big(s^{-1+\alpha}\varphi_0(s)\Big)\prod_{j=1}^2\Big(\frac{(\widetilde{F}^{\pm})'(r_js)}{r_js}\sin^2\alpha_j+(\widetilde{F}^{\pm})^{(2)}(r_js)\cos^2\alpha_j\Big)ds\Big| \\
&+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-1+\alpha}\varphi_0(s)\partial_s\prod_{j=1}^2\Big(\frac{(\widetilde{F}^{\pm})'(r_js)}{r_js}\sin^2\alpha_j+(\widetilde{F}^{\pm})^{(2)}(r_js)\cos^2\alpha_j\Big)ds\Big|\Big) \\
:=&\frac{2^{(3+\alpha)N}}{1+|t|2^{4N}}\Big(|\mathcal{E}^{2,\pm}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|+|\mathcal{E}^{2,\pm}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)|\Big). \qquad (3.32)
\end{aligned}
$$

We first estimate the term $\mathcal{E}^{2,\pm}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)$. Let

$$
\partial_s\prod_{j=1}^2\Big(\frac{(\widetilde{F}^{\pm})'(r_js)}{r_js}\sin^2\alpha_j+(\widetilde{F}^{\pm})^{(2)}(r_js)\cos^2\alpha_j\Big):=e^{\pm ir_1s}e^{\pm ir_2s}s^{-1}\widetilde{F}^{\pm}_{\alpha_1,\alpha_2}(r_1s,r_2s),
$$

then

$$
\begin{aligned}
\big|\mathcal{E}^{2,\pm}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\big|\lesssim&\Big|\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x-\theta_2u_2|+|y-\theta_1u_1|)}s^{-2+\alpha}\varphi_0(s) \\
&\times\widetilde{F}^{\pm}_{\alpha_1,\alpha_2}(2^Ns|x-\theta_2u_2|,2^Ns|y-\theta_1u_1|)\Big|.
\end{aligned}
$$

Note that

$$
\big|\partial_s^k\widetilde{F}^{\pm}_{\alpha_1,\alpha_2}(2^Ns|x-\theta_2u_2|,2^Ns|y-\theta_1u_1|)\lesssim 1,\quad k=0,1,
$$

by Lemma 2.1 with $z=(x,y,\theta_1,\theta_2,u_1,u_2)$ and

$$
\Psi(z)=|x-\theta_2u_2|+|y-\theta_1u_1|,\quad \Phi(2^Ns,z)=\widetilde{F}^{\pm}_{\alpha_1,\alpha_2}(2^Ns|x-\theta_2u_2|,2^Ns|y-\theta_1u_1|),
$$

we obtain that $\mathcal{E}^{2,\pm}_{2,N}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. Similarly, we get that $\mathcal{E}^{2,\pm}_{1,N}$ is controlled by the same bound. Hence $\widetilde{E}^{2,\pm}_{1,N}$ is bounded by by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. By (3.32) and Hölder's inequality we obtain that $\widetilde{K}^{2,\pm}_{1,N}$ is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$. By the same summing way with the proof of (2.10), we immediately obtain (3.30a).

For the term $\Gamma^2_{-3,2}(\lambda)$. We use the orthogonality $S_2x_iv=0$, $i=1,2,3$ for the left hand of $\Gamma^2_{-3,2}(\lambda)$ and $S_2v=0$ for the right hand of $\Gamma^2_{-3,2}(\lambda)$, by the same argument with the proof of the term $\Gamma^2_{-3,1}(\lambda)$, we immediately obtain the desired conclusion. Similarly, we also get the desired integral estimate for the term $\Gamma^2_{-3,3}(\lambda)$. □

Proposition 3.6. *Assume that* $|V(x)|\lesssim(1+|x|)^{-19-}$. *Let* $\Gamma^2_{-3,4}(\lambda)$ *be an operator defined in* (3.29). *Suppose that*

$$
K_{\alpha,t}(x,y):=\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-3,4}(\lambda)(x,y)d\lambda, \qquad (3.33)
$$

is the integral kernel of the operator $K_{\alpha,t}$. Then for any $0\leq\alpha\leq 3$,

$$\sup_{x,y\in\mathbb{R}^3}\big|K_{\alpha,t}(x,y)\big|\lesssim|t|^{-\frac{1+\alpha}{4}}.$$

In particular, if $G_0vS_2A^2_{-3,1}S_2vG_0\neq 0$, then

$$\|K_{\alpha,t}\|_{L^1\to L^\infty}\sim|t|^{-\frac{1+\alpha}{4}},\quad |t|>1.$$

Proof. Note that

$$\Gamma^2_{-3,4}(\lambda)=G_0v(\lambda^{-3}S_2A^2_{-3,1}S_2)vG_0 \quad\text{ and }\quad G_0=-\frac{|x-y|}{8\pi},$$

then

$$\begin{aligned}K_{\alpha,t}(x,y)=&\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^\alpha[G_0vS_2A^2_{-3,1}S_2vG_0](x,y)d\lambda\\ =&\frac{1}{64\pi^2}\Big(\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^\alpha d\lambda\Big)\int_{\mathbb{R}^6}|x-u_1|v(u_1)(S_2A^2_{-3,1}S_2)(u_1,u_2)v(u_2)\\ &\times|y-u_2|du_1du_2.\end{aligned}$$

Referring to Stein's book [23, p. 356], one has

$$\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^\alpha d\lambda\Big|\sim|t|^{-\frac{1+\alpha}{4}},\quad |t|>1.$$

By the orthogonality $S_2v=0$ and Hölder's inequality, we obtain that

$$\begin{aligned}&\big|G_0vS_2A^2_{-3,1}S_2vG_0(x,y)\big|\\ =&\frac{1}{64\pi^2}\Big|\int_{\mathbb{R}^6}(|x-u_1|-|x|)v(u_1)(S_2A^2_{-3,1}S_2)(u_1,u_2)v(u_2)\times(|y-u_2|-|y|)du_1du_2\Big|\\ \lesssim&\big\||u_1|v(u_1)\|_{L^2}\big\|S_2A^2_{-3,1}S_2\|_{L^2\to L^2}\big\||u_2|v(u_2)\big\|_{L^2}\lesssim 1.\end{aligned}$$

Hence,

$$\sup_{x,y\in\mathbb{R}^3}\big|K_{\alpha,t}(x,y)\big|\lesssim|t|^{-\frac{1+\alpha}{4}},$$

which gives

$$\|K_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha}{4}}.$$

Next, assume that there exists $\sigma>0$ such that

$$\|K_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha+\sigma}{4}},\quad |t|>1.$$

According to the hypothetical condition $G_0vS_2A^2_{-3,1}S_2vG_0\neq 0$, there exist $f,g\in L^1(\mathbb{R}^3)$ with $\|f\|_{L^1}=\|g\|_{L^1}=1$ and constant $D>0$ such that

$$\big|\langle K_{\alpha,t}f,g\rangle\big|=\Big|\Big(\int_0^\infty \chi(\lambda)e^{-it\lambda^4}\lambda^\alpha d\lambda\Big)\cdot\Big\langle G_0vS_2A^2_{-3,1}S_2vG_0f,g\Big\rangle\Big|>D|t|^{-\frac{1+\alpha}{4}}.$$

Then

$$D|t|^{-\frac{1+\alpha}{4}}\lesssim|t|^{-\frac{1+\alpha+\sigma}{4}}\quad\Rightarrow\quad D\lesssim|t|^{-\frac{\sigma}{4}}.$$

As a result, we obtain that $D=0$ as $|t|\to\infty$, which is a contradiction. Hence,

$$\|K_{\alpha,t}\|_{L^1\to L^\infty}\sim|t|^{-\frac{1+\alpha}{4}},\quad |t|>1.$$

Thus the proof of this proposition is completed. □

Secondly, we estimate the second term $\Omega_{2,2}(\lambda)$ in (3.27). We have

$$\begin{aligned}\Omega_{2,2}(\lambda)=&(R_0^\pm(\lambda^4)-G_0)v(\lambda^{-2}S_2A^2_{-2,1}S_1)vR_0^\pm(\lambda^4)+G_0v(\lambda^{-2}S_2A^2_{-2,1}S_1)vR_0^\pm(\lambda^4)\\ :=&\Gamma^2_{-2,1}(\lambda)+\Gamma^2_{-2,2}(\lambda).\end{aligned}\tag{3.34}$$

Proposition 3.7. *Assume that* $|V(x)|\lesssim(1+|x|)^{-19-}$. *Let* $\Gamma^2_{-2,j}(\lambda)$ $(j=1,2)$ *be a family of operators defined in* (3.34). *Then*

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-2,1}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}},\qquad 0\le\alpha\le3,\tag{3.35a}$$

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-2,2}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+\alpha}{4}},\qquad 0\le\alpha\le3.\tag{3.35b}$$

Proof. By Lemma 3.2(i) and (ii), using the same method with the proof of Proposition 3.5, we obtain (3.35a). By using Lemma 3.2(i), by the same arguments with the proof of Proposition 3.2, we immediately obtain (3.35b). □

Finally, we deal with the term $\Omega_{2,3}(\lambda)$ in (3.27). We have

$$\begin{aligned}\Omega_{2,3}(\lambda)=&(R_0^\pm(\lambda^4)-G_0)v(\lambda^{-1}S_2A^2_{-1,1})vR_0^\pm(\lambda^4)+G_0v(\lambda^{-1}S_2A^2_{-1,1})vR_0^\pm(\lambda^4)\\ :=&\Gamma^2_{-1,1}(\lambda)+\Gamma^2_{-1,2}(\lambda).\end{aligned}\tag{3.36}$$

Similar to the proof of Proposition 3.7, we obtain the following proposition.

Proposition 3.8. *Assume that* $|V(x)|\lesssim(1+|x|)^{-19-}$. *Let* $\Gamma^2_{-1,j}(\lambda)$ $(j=1,2)$ *be operators defined in* (3.36). *Then*

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-1,1}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}},\qquad 0\le\alpha\le3,\tag{3.37a}$$

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\Gamma^2_{-1,2}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+\alpha}{4}},\qquad 0\le\alpha\le3.\tag{3.37b}$$

Combining with Propositions 3.5-3.8 and the proof of the first kind resonance, we immediately obtain that for $0\le\alpha\le3$,

$$\big\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)\big\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{2+\alpha}{4}}.$$

Let $F_{\alpha,t}$ be an operator with the integral kernel

$$\begin{aligned}F_{\alpha,t}(x,y)=K_{\alpha,t}(x,y)+\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[&\Gamma^2_{-3,2}(\lambda)+\Gamma^2_{-3,3}(\lambda)\\&+\Gamma^2_{-2,2}(\lambda)+\Gamma^2_{-1,2}(\lambda)\big](x,y)d\lambda,\end{aligned}\tag{3.38}$$

where $K_{\alpha,t}$, $\Gamma^2_{-3,i}(\lambda)$ $(i=2,3)$, $\Gamma^2_{-2,2}(\lambda)$ and $\Gamma^2_{-1,2}(\lambda)$ are operators defined in (3.33), (3.29), (3.34) and (3.36). By using the estimates (3.30b), (3.33), (3.35b) and (3.37b), we obtain that

$$F_{\alpha,t}=K_{\alpha,t}+\mathcal{O}\big(|t|^{-\frac{2+\alpha}{4}}\big),\quad 0\le\alpha\le3,\tag{3.39}$$

and

$$\|F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha}{4}},\quad 0\le\alpha\le3.\tag{3.40}$$

Combining with Propositions 3.5-3.8 and the proof of the first kind resonance again, we immediately obtain that for $0\le\alpha\le3$,

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)-F_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\tag{3.41}$$

which gives (3.14).

Hence the proof of Theorem 3.2(iii) in the second kind resonance is completed.

3.2.4 The third kind of resonance

If zero is the third kind of resonance of H, then using (3.3) and (3.9) one has

$$\begin{aligned}R_V^\pm(\lambda^4)=R_0^\pm(\lambda^4)-R_0^\pm(\lambda^4)v\Big(\lambda^{-4}S_3D_3S_3\Big)vR_0^\pm(\lambda^4)&\\-R_0^\pm(\lambda^4)v\Big(\lambda^{-3}S_2A^3_{-3,1}S_2\Big)vR_0^\pm(\lambda^4)&\\-R_0^\pm(\lambda^4)v\Big(\lambda^{-2}S_2A^3_{-2,1}S_1+\lambda^{-2}S_1A^3_{-2,2}S_2\Big)vR_0^\pm(\lambda^4)&\\-R_0^\pm(\lambda^4)v\Big(\lambda^{-1}S_2A^3_{-1,1}+\lambda^{-1}A^3_{-1,2}S_2+\lambda^{-1}S_1A^3_{-1,3}S_1\Big)vR_0^\pm(\lambda^4)&\\-R_0^\pm(\lambda^4)v\Big(S_1A^3_{0,1}+A^3_{0,2}S_1+QA^3_{0,3}Q\Big)vR_0^\pm(\lambda^4)&\\-R_0^\pm(\lambda^4)v\Gamma_1(\lambda)vR_0^\pm(\lambda^4).&\end{aligned}\tag{3.42}$$

In order to prove Theorem 3.2(iii) in the third kind resonance, we need to analyze what contribution the term $\lambda^{-4}S_3D_3S_3$ has in Stone's formula (3.10). By a simple calculation, we obtain that

$$\begin{aligned}&R_0^+(\lambda^4)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)-R_0^-(\lambda^4)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^-(\lambda^4)\\ =&\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)\\ &+R_0^-(\lambda^4)v\big(\lambda^{-4}S_3D_3S_3\big)v\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big).\end{aligned}$$

Proposition 3.9. *Let $|V(x)|\lesssim(1+|x|)^{-23-}$. Then for any $x,y\in\mathbb{R}^3$ and $0\le\alpha\le 3$,*

$$\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+2\alpha}{4}},$$
$$\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^-(\lambda^4)v\big(\lambda^{-4}S_3D_3S_3\big)v\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+2\alpha}{4}}.$$

Proof. We only estimate the first integral, similarly for the second integral. We write

$$\begin{aligned}K_{1,N}^{3,\pm}(t;x,y):=\int_0^\infty &e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)\\ &\cdot v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)\big](x,y)d\lambda.\end{aligned}\tag{3.43}$$

Let

$$\bar{R}_0(p)=\frac{e^{ip}-e^{-ip}}{p}\quad\text{and}\quad F^\pm(p)=\frac{e^{\pm ip}-e^{-p}}{p}.$$

Then

$$R_0^+(\lambda^4)-R_0^-(\lambda^4)=\frac{1}{8\pi\lambda}\bar{R}_0(\lambda|x-y|),\quad R_0^\pm(\lambda^4)=\frac{1}{8\pi\lambda}F^\pm(\lambda|x-y|).$$

Note that $\bar{R}_0\in C^5(\mathbb{R})$ and $(\bar{R}_0)'(p)=0$, but

$$(\bar{R}_0)^{(2)}(p)=-\frac{2i}{3}\neq 0.$$

Let

$$\bar{F}(p)=\bar{R}_0(p)+\frac{2i}{3}p^2,\quad \bar{F}(p)=\frac{e^{ip}-e^{-ip}}{p}+\frac{ip^2}{3}.$$

Hence, $\bar{F}\in C^5(\mathbb{R})$ and $(\bar{F})^{(k)}(p)=0,\ k=1,2$. By the orthogonality $S_3v=S_3x_iv=$

$S_3x_ix_jv=0$ $(i,j=1,2,3)$, one has

$$\begin{aligned}
&\big[\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)\big](x,y)\\
=&\frac{1}{64\pi^2\lambda^6}\int_{\mathbb{R}^6}\bar{R}_0(\lambda|x-u_2|)v(u_2)(S_3D_3S_3)(u_2,u_1)v(u_1)F^+(\lambda|y-u_1|)du_1du_2\\
=&\frac{1}{64\pi^2\lambda^6}\int_{\mathbb{R}^6}\big(\bar{R}_0(\lambda|x-u_2|)+\frac{2i}{3}\lambda^2|x-u_2|^2\big)v(u_2)(S_3D_3S_3)(u_2,u_1)v(u_1)\\
&\times F^+(\lambda|y-u_1|)du_1du_2\\
=&\frac{1}{64\pi^2\lambda^6}\int_{\mathbb{R}^6}\bar{F}(\lambda|x-u_2|)v(u_2)(S_3D_3S_3)(u_2,u_1)v(u_1)F^+(\lambda|y-u_1|)du_1du_2.
\end{aligned}$$

By Lemma 3.2(iii), one has

$$\begin{aligned}
&\big[\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v\big(\lambda^{-4}S_3D_3S_3\big)vR_0^+(\lambda^4)\big](x,y)\\
=&-\frac{1}{128\pi^2\lambda^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1(1-\theta_2)^2\Big[\Big(\frac{\bar{F}'(\lambda|x-\theta_2u_2|)}{\lambda^2|x-\theta_2u_2|^2}-\frac{\bar{F}^{(2)}(\lambda|x-\theta_2u_2|)}{\lambda|x-\theta_2u_2|}\Big)3\cos\alpha_2\sin^2\alpha_2\\
&-\bar{F}^{(3)}(\lambda|x-\theta_2u_2|)\cos^3\alpha_2\Big](F^+)'(\lambda|y-\theta_1u_1|)\cos\alpha_1d\theta_1\theta_2\\
&\times|u_2|^3v(u_2)v(u_1)|u_1|(S_3D_3S_3)(u_2,u_1)du_1du_2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&K_{1,N}^{3,\pm}(t;x,y)\\
=&-\frac{1}{128\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1\Big[\int_0^\infty e^{-it\lambda^4}\lambda^{1+\alpha}\varphi_0(2^{-N}\lambda)\Big[\Big(\frac{\bar{F}'(\lambda|x-\theta_2u_2|)}{\lambda^2|x-\theta_2u_2|^2}-\frac{\bar{F}^{(2)}(\lambda|x-\theta_2u_2|)}{\lambda|x-\theta_2u_2|}\Big)\\
&\times3\cos\alpha_2\sin^2\alpha_2-\bar{F}^{(3)}(\lambda|x-\theta_2u_2|)\cos^3\alpha_2\Big](F^+)'(\lambda|y-\theta_1u_1|)\cos\alpha_1d\lambda\Big](1-\theta_2)^2d\theta_1\theta_2\\
&\times|u_2|^3|u_1|v(u_2)v(u_1)(S_3D_3S_3)(u_2,u_1)du_1du_2\\
:=&-\frac{1}{128\pi^2}\int_{\mathbb{R}^6}\int_0^1\int_0^1E_{1,N}^{3,+}(t;x,y,\theta_1,\theta_2,u_1,u_2)(1-\theta_2)^2d\theta_1\theta_2|u_2|^3|u_1|v(u_2)v(u_1)\\
&\times(S_3D_3S_3)(u_2,u_1)du_1du_2.
\end{aligned}\tag{3.44}$$

Let $s=2^{-N}\lambda$, then

$$\begin{aligned}
E_{1,N}^{3,+}=&2^{(2+\alpha)N}\int_0^\infty e^{-it2^{4N}s^4}s^{1+\alpha}\varphi_0(s)\Big[\Big(\frac{\bar{F}'(2^Ns|x-\theta_2u_2|)}{(2^Ns)^2|x-\theta_2u_2|^2}-\frac{\bar{F}^{(2)}(2^Ns|x-\theta_2u_2|)}{2^Ns|x-\theta_2u_2|}\Big)\\
&\times3\cos\alpha_2\sin^2\alpha_2-\bar{F}^{(3)}(2^Ns|x-\theta_2u_2|)\cos^3\alpha_2\Big](F^+)'(2^Ns|y-\theta_1u_1|)\cos\alpha_1ds.
\end{aligned}$$

Let

$$r_1=2^N|y-\theta_1u_1| \quad \text{and} \quad r_2=2^N|x-\theta_2u_2|.$$

Then by using integration by parts,

$$\begin{aligned}|E^{3,+}_{1,N}\lesssim&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\Big(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\big(s^{-2+\alpha}\varphi_0(s)\big)\Big(\Big(\frac{\bar F'(r_2s)}{(r_2s)^2}-\frac{\bar F^{(2)}(r_2s)}{r_2s}\Big)3\cos\alpha_2\sin^2\alpha_2\\&-\bar F^{(3)}(r_2s)\cos^3\alpha_2\Big)(F^+)'(r_1s)\cos\alpha_1ds\Big|+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\\&\times\partial_s\Big[\Big(\Big(\frac{\bar F'(r_2s)}{(r_2s)^2}-\frac{\bar F^{(2)}(r_2s)}{r_2s}\Big)3\cos\alpha_2\sin^2\alpha_2-\bar F^{(3)}(r_2s)\cos^3\alpha_2\Big)(F^+)'(r_1s)\cos\alpha_1\Big]ds\Big|\Big)\\:=&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\Big(\big|\mathcal{E}^{3,+}_{1,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\big|+\big|\mathcal{E}^{3,+}_{2,N}(t;x,y,\theta_1,\theta_2,u_1,u_2)\big|\Big).\end{aligned}$$

For $\mathcal{E}^{3,+}_{2,N}$. Note that

$$\begin{aligned}&\partial_s\Big[\Big(\Big(\frac{\bar F'(r_2s)}{(r_2s)^2}-\frac{\bar F^{(2)}(r_2s)}{r_2s}\Big)3\cos\alpha_2\sin^2\alpha_2-\bar F^{(3)}(r_2s)\cos^3\alpha_2\Big)(F^+)'(r_1s)\cos\alpha_1\Big]\\:=&e^{ir_1s}e^{ir_2s}s^{-1}\bar F_{\alpha_1,\alpha_2}(r_1s,r_2s),\end{aligned}$$

then we have

$$\big|\mathcal{E}^{3,+}_{2,N}\big|\lesssim\Big|\int_0^\infty e^{-it2^{4N}s^4}e^{i(r_1+r_2)s}s^{-3+2\alpha}\varphi_0(s)\bar F_{\alpha_1,\alpha_2}(r_1s,r_2s)ds\Big|.$$

Since

$$\big|\partial_s^k\bar F_{\alpha_1,\alpha_2}(2^Ns|x-\theta_2u_2|,\quad 2^Ns|y-\theta_1u_1|)\big|\lesssim 1,\quad k=0,1,$$

by Lemma 2.1 with $z=(x,y,\theta_1,\theta_2,u_1,u_2)$ and

$$\begin{aligned}&\Psi(z)=r_1+r_2=|x-\theta_2u_2|+|y-\theta_1u_1|,\\&\Phi(2^Ns,z)=\bar F_{\alpha_1,\alpha_2}(2^Ns|x-\theta_2u_2|,2^Ns|y-\theta_1u_1|),\end{aligned}$$

we obtain that $\mathcal{E}^{3,+}_{2,N}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. Similar to obtain that $\mathcal{E}^{3,+}_{1,N}$ is controlled by the same bound. Hence $E^{3,+}_{1,N}$ is bounded by $2^{(2+\alpha)N}\Theta_{N_0,N}(t)$. By (3.44) and Hölder's inequality we obtain that $K^{3,\pm}_{1,N}$ is bounded by $2^{(2+\alpha)N}\Theta_{N_0,N}(t)$. □

In the proof of Proposition 3.9, for the projection operator S_3 in the right hand of the integral (3.43), we only use the orthogonality $S_3v=0$. As a result, we don't get the same decay estimate as in the regular case. Similarly in the second kind of resonance case, we will subtract a specific operator to get the same decay estimate rate as the regular case. Note that

$$\begin{aligned}
&R_0^+(\lambda^4)v(\lambda^{-4}S_3D_3S_3)vR_0^+(\lambda^4)-R_0^-(\lambda^4)v(\lambda^{-4}S_3D_3S_3)vR_0^-(\lambda^4)\\
=&\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v(\lambda^{-4}S_3D_3S_3)vR_0^+(\lambda^4)\\
&+R_0^-(\lambda^4)v(\lambda^{-4}S_3D_3S_3)v\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)\\
=&\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)v(\lambda^{-4}S_3D_3S_3)v\big(R_0^+(\lambda^4)-G_0\big)\\
&+(R_0^+(\lambda^4)-R_0^-(\lambda^4))v(\lambda^{-4}S_3D_3S_3)vG_0\\
&+\big(R_0^-(\lambda^4)-G_0\big)v(\lambda^{-4}S_3D_3S_3)v\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)\\
&+G_0v(\lambda^{-4}S_3D_3S_3)v\big(R_0^+(\lambda^4)-R_0^-(\lambda^4)\big)\\
:=&\Gamma^3_{-4,1}(\lambda)+\Gamma^3_{-4,2}(\lambda)+\Gamma^3_{-4,3}(\lambda)+\Gamma^3_{-4,4}(\lambda).
\end{aligned}\tag{3.45}$$

Hence, in order to complete the proof of Theorem 3.2(iii) in the third kind resonnace, combining with the analysis of the second kind of resonance case, it suffices to show the following proposition.

Proposition 3.10. *Assume that* $|V(x)|\lesssim(1+|x|)^{-23-}$. *Let* $\Gamma^3_{-3,j}(\lambda)$ $(j=1,\cdots,4)$ *be operators defined in* (3.45). *Then*

$$\begin{aligned}
&\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+2\alpha}\Gamma^3_{-4,j}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+2\alpha}{4}}, \qquad 0\le\alpha\le3, \quad j=1,3,\\
&\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+2\alpha}\Gamma^3_{-4,j}(\lambda)(x,y)d\lambda\Big|\lesssim|t|^{-\frac{2+2\alpha}{4}}, \qquad 0\le\alpha\le3, \quad j=2,4.
\end{aligned}$$

Proof. By the same argument with the proof of Proposition 3.5 and Proposition 3.9, we obtain that this proposition holds. Here we omit these details. □

We write

$$\widetilde{F}_{\alpha,t}(x,y)=F_{\alpha,t}(x,y)+\int_0^\infty\chi(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[\Gamma^3_{-3,1}(\lambda)+\Gamma^3_{-3,3}(\lambda)\big](x,y)d\lambda.$$

By using the estimate (3.40) and Proposition 3.10, we obtain that

$$\|\widetilde{F}_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{1+\alpha}{4}}, \quad 0\le\alpha\le3.$$

Combining with Propositions 3.5-3.10 and the proof of the second kind resonance again, we immediately obtain that for $0\le\alpha\le3$,

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\chi(H)-\widetilde{F}_{\alpha,t}\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}}.$$

Thus the proof of Theorem 3.2(iii) is completed. □

4 High energy dispersive estimates

In this subsection, we are devoted to establishing the decay bounds of Theorem 1.1 for high energy. Furthermore, it suffices to prove the following theorem.

Theorem 4.1 (High Energy Dispersive Estimate). *Let* $|V(x)|\lesssim\langle x\rangle^{-4-}$. *Assume that* $H=\Delta^2+V(x)(x\in\mathbb{R}^3)$ *has no positive embedding eigenvalues and* $P_{ac}(H)$ *denotes the projection onto absolutely continuous spectrum space of* H. *Then*

$$\|H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\widetilde{\chi}(H)\|_{L^1\to L^\infty}\lesssim|t|^{-\frac{3+\alpha}{4}},\quad 0\leq\alpha\leq 3. \tag{4.1}$$

To complete the proof of Theorem 4.1, we will use the following Stone's formula,

$$H^{\frac{\alpha}{4}}e^{-itH}P_{ac}(H)\widetilde{\chi}(H)f=\sum_{N=N_0'+1}^{+\infty}\sum_{\pm}\frac{2}{\pi i}\int_0^\infty e^{-it\lambda^4}\varphi_0(2^{-N}\lambda)\lambda^{3+\alpha}R_V^{\pm}(\lambda^4)fd\lambda, \tag{4.2}$$

and the resolvent identity,

$$R_V^{\pm}(\lambda^4)=R_0^{\pm}(\lambda^4)-R_0^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)+R_0^{\pm}(\lambda^4)VR_V^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4). \tag{4.3}$$

Combining with Proposition 2.1, it is enough to establish the following Propositions 4.1 and 4.2.

Proposition 4.1. *Assume that* $|V(x)|\lesssim(1+|x|)^{-3-}$. *Let* $\Theta_{N_0,N}(t)$ *be a function defined in* (2.2) *and* $N>N'$. *Then for each* x, y *and* $0\leq\alpha\leq 3$,

$$\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t).$$

Moreover,

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\widetilde{\chi}(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}}.$$

Proof. We write

$$L_{1,N}^{\pm}(t;x,y):=\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda.$$

Then

$$\int_0^\infty\widetilde{\chi}(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda=\sum_{N=N'+1}^{\infty}L_{1,N}^{\pm}(t;x,y).$$

Let $F^{\pm}(p)=\frac{e^{\pm ip}-e^{-p}}{p}$, then

$$R_0^{\pm}(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^{\pm}(\lambda|x-y|).$$

Set $s=2^{-N}\lambda$, one has

$$\begin{aligned}
&L_{1,N}^{\pm}(t;x,y)\\
=&\int_{\mathbb{R}^3}\int_0^{\infty}e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)R_0^{\pm}(\lambda^4)(x,u_1)V(u_1)R_0^{\pm}(\lambda^4)(u_1,y)d\lambda du_1\\
=&\frac{2^{(2+\alpha)N}}{64\pi^2}\int_{\mathbb{R}^3}\int_0^{\infty}e^{-it2^{4N}s^4}s^{1+\alpha}\varphi_0(s)F^{\pm}(2^Ns|x-u_1|)F^{\pm}(2^Ns|y-u_1|)dsV(u_1)du_1\\
:=&\frac{1}{64\pi^2}\int_{\mathbb{R}^3}E_{1,N}^{\pm}(t;x,y,u_1)V(u_1)du_1.
\end{aligned}$$

By using integration by parts, we have

$$\begin{aligned}
|E_{1,N}^{\pm}|\lesssim&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\bigg(\Big|\int_0^{\infty}e^{-it2^{4N}s^4}\partial_s\big(s^{-2+\alpha}\varphi_0(s)\big)F^{\pm}(2^Ns|x-u_1|)F^{\pm}(2^Ns|y-u_1|)ds\Big|\\
&\qquad+\Big|\int_0^{\infty}e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\partial_s\Big(F^{\pm}(2^Ns|x-u_1|)F^{\pm}(2^Ns|y-u_1|)\Big)ds\Big|\bigg)\\
:=&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\Big(|\mathcal{E}_{1,N}^{\pm}(t;x,y,u_1)|+|\mathcal{E}_{2,N}^{\pm}(t;x,y,u_1)|\Big).
\end{aligned}\tag{4.4}$$

For $\mathcal{E}_{2,N}^{\pm}$. Since

$$\begin{aligned}
&\partial_s\Big(F^{\pm}(2^Ns|x-u_1|)F^{\pm}(2^Ns|y-u_1|)\Big)\\
:=&s^{-1}e^{\pm i2^Ns(|x-u_1|+|y-u_1|)}\\
&\times\Big(F_1^{\pm}(2^Ns|x-u_1|)F_0^{\pm}(2^Ns|y-u_1|)+F_0^{\pm}(2^Ns|x-u_1|)F_1^{\pm}(2^Ns|y-u_1|)\Big),
\end{aligned}$$

where

$$F_1^{\pm}(p)=pe^{\mp ip}(F^{\pm})'(p)=\frac{(\pm ip-1)+(p+1)e^{-p\mp ip}}{p},\quad F_0^{\pm}(p)=\frac{1-e^{-p\mp ip}}{p}.$$

Then we have

$$\begin{aligned}
|\mathcal{E}_{2,N}^{\pm}|\lesssim&\Big|\int_0^{\infty}e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x-u_1|+|y-u_1|)}s^{-3+2\alpha}\varphi_0(s)\Big(F_1^{\pm}(2^Ns|x-u_1|)\\
&\times F_0^{\pm}(2^Ns|y-u_1|)+F_0^{\pm}(2^Ns|x-u_1|)F_1^{\pm}(2^Ns|y-u_1|)\Big)ds\Big|.
\end{aligned}$$

Since $N>N_0'$, then

$$
\begin{aligned}
|\mathcal{E}_{2,N}^{\pm}(t;x,y)|\lesssim \Big|\int_0^\infty & e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x-u_1|+|y-u_1|)}s^{-3+2\alpha}\varphi_0(s)\Big(F_1^{\pm}(2^Ns|x-u_1|)\\
&\times F_0^{\pm}(2^Ns|y-u_1|)+F_0^{\pm}(2^Ns|x-u_1|)F_1^{\pm}(2^Ns|y-u_1|)\Big)ds\Big|.
\end{aligned}
$$

Noting that for $k=0,1$,

$$
\begin{aligned}
&\big|\partial_s^k\big(F_1^{\pm}(2^Ns|x-u_1|)F_0^{\pm}(2^Ns|y-u_1|)\big)\big|\lesssim 1,\\
&\big|\partial_s^k\big(F_0^{\pm}(2^Ns|x-u_1|)F_1^{\pm}(2^Ns|y-u_1|)\big)\big|\lesssim 1,
\end{aligned}
$$

then by Lemma 2.1 with $z=(x,y,u_1)$, $\Psi(z)=|x-u_1|+|y-u_1|$, and

$$
\Phi(2^Ns;z)=\Big(F_1^{\pm}(2^Ns|x-u_1|)F_0^{\pm}(2^Ns|y-u_1|)+F_0^{\pm}(2^Ns|x-u_1|)F_1^{\pm}(2^Ns|y-u_1|)\Big),
$$

we obtain that $\mathcal{E}_{2,N}^{\pm}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. Similarly, $\mathcal{E}_{1,N}^{\pm}$ is controlled by the same bound. By (4.4), we obtain that $E_{1,N}^{\pm}$ is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$. Hence we have

$$
|L_{1,N}^{\pm}(t;x,y)|\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t)\int_{\mathbb{R}^3}|V(u_1)|du_1\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t).
$$

Finally, by the same summing way with the proof of (2.10), we immediately obtain the desired conclusion. □

In order to deal with the term $R_0^{\pm}(\lambda^4)VR_V^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)$, we need to give a lemma as follows, see [10].

Lemma 4.1. *Let $k\geq 0$ and $|V(x)|\lesssim(1+|x|)^{-k-1-}$ such that $H=\Delta^2+V$ has no embedded positive eigenvalues. Then for any $\sigma>k+\frac{1}{2}$, $R_V^{\pm}(\lambda)\in\mathcal{B}\big(L^2_\sigma(\mathbb{R}^3),L^2_{-\sigma}(\mathbb{R}^3)\big)$ are C^k-continuous for all $\lambda>0$. Furthermore,*

$$
\big\|\partial_\lambda^k R_V^{\pm}(\lambda)\big\|_{L^2_\sigma(\mathbb{R}^3)\to L^2_{-\sigma}(\mathbb{R}^3)}=\mathcal{O}\big(|\lambda|^{\frac{-3(k+1)}{4}}\big)\quad \text{as}\quad \lambda\to+\infty.
$$

Proposition 4.2. *Assume that $|V(x)|\lesssim(1+|x|)^{-4-}$. Let $\Theta_{N_0,N}(t)$ be a function defined in (2.2) and $N>N'$. Then for each x,y and $0\leq\alpha\leq 3$,*

$$
\begin{aligned}
&\Big|\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\big[R_0^{\pm}(\lambda^4)VR_V^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\\
&\lesssim 2^{(3+\alpha)N}\Theta_{N_0,N}(t). \qquad (4.5)
\end{aligned}
$$

Moreover,

$$
\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty\widetilde{\chi}(\lambda)e^{-it\lambda^4}\lambda^{3+\alpha}\big[R_0^{\pm}(\lambda^4)VR_V^{\pm}(\lambda^4)VR_0^{\pm}(\lambda^4)\big](x,y)d\lambda\Big|\lesssim|t|^{-\frac{3+\alpha}{4}}. \qquad (4.6)
$$

Proof. In order to get (4.5), it's equivalent to prove that

$$L^{\pm}_{2,N}(t;x,y):=\int_0^\infty e^{-it\lambda^4}\lambda^{3+\alpha}\varphi_0(2^{-N}\lambda)\Big\langle VR^{\pm}_V(\lambda^4)V\Big(R^{\pm}_0(\lambda^4)(*,y)\Big)(\cdot), \Big(R^{\pm}_0(\lambda^4)\Big)^*(x,\cdot)\Big\rangle d\lambda$$

is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$.

In fact, let $F^{\pm}(p)=\frac{e^{\pm ip}-e^{-p}}{p}$, then

$$R^{\pm}_0(\lambda^4)(x,y)=\frac{1}{8\pi\lambda}F^{\pm}(\lambda|x-y|).$$

Hence

$$\begin{aligned}&\Big\langle VR^{\pm}_V(\lambda^4)V(R^{\pm}_0(\lambda^4)(*,y))(\cdot),\ R^{\mp}_0(\lambda)(x,\cdot)\Big\rangle\\ =&\frac{1}{64\pi^2\lambda^2}\Big\langle VR^{\pm}_V(\lambda^4)V\Big(F^{\pm}(\lambda|*-y|)\Big)(\cdot),\ F^{\mp}(\lambda|x-\cdot|)\Big\rangle\\ :=&\frac{1}{64\pi^2\lambda^2}E^{L,\pm}_2(\lambda;x,y).\end{aligned}$$

Let $s=2^{-N}\lambda$, then

$$L^{\pm}_{2,N}(t;x,y):=\frac{2^{(2+\alpha)N}}{64\pi^2}\int_0^\infty e^{-it2^{4N}s^4}s^{1+\alpha}\varphi_0(s)E^{L,\pm}_2(2^Ns;x,y)ds.$$

By using integration by parts, we have

$$\begin{aligned}|L^{\pm}_{2,N}(t;x,y)|\lesssim&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\bigg(\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\big(s^{-2+\alpha}\varphi_0(s)\big)\partial_sE^{L,\pm}_2(2^Ns;x,y)ds\Big|\\ &+\Big|\int_0^\infty e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\partial_s\big(E^{L,\pm}_2(2^Ns;x,y)\big)ds\Big|\bigg)\\ :=&\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\Big(|\mathcal{E}^{L,\pm}_{1,N}(t;x,y)|+|\mathcal{E}^{L,\pm}_{2,N}(t;x,y)|\Big).\end{aligned}\tag{4.7}$$

For $\mathcal{E}^{L,\pm}_{2,N}$. Since

$$\begin{aligned}\partial_s\big(E^{L,\pm}_2(2^Ns;x,y)\big)=&\Big\langle V\partial_s\big(R_V(2^Ns^4)\big)V\big(F^{\pm}(2^Ns|*-y|)\big)(\cdot),\ F^{\mp}(2^Ns|x-\cdot|)\Big\rangle\\ &+\Big\langle VR_V(2^Ns^4)V\big(\partial_sF^{\pm}(2^Ns|*-y|)\big)(\cdot),\ F^{\mp}(2^Ns|x-\cdot|)\Big\rangle\\ &+\Big\langle VR_V(2^Ns^4)V\big(F^{\pm}(2^Ns|*-y|)\big)(\cdot),\ \partial_sF^{\mp}(2^Ns|x-\cdot|)\Big\rangle\\ :=&E^{L,\pm}_{21}(2^Ns;x,y)+E^{L,\pm}_{22}(2^Ns;x,y)+E^{L,\pm}_{23}(2^Ns;x,y),\end{aligned}$$

then we have

$$\begin{aligned}\mathcal{E}^{L,\pm}_{2,N}(t;x,y)=&\int_0^\infty e^{-it2^{4N}s^4}s^{-2+\alpha}\varphi_0(s)\big(E^{L,\pm}_{21}+E^{L,\pm}_{22}+E^{L,\pm}_{23}\big)(2^Ns;x,y)ds\\ :=&\mathcal{E}^{\pm,L}_{21,N}(t;x,y)+\mathcal{E}^{\pm,L}_{22,N}(t;x,y)+\mathcal{E}^{\pm,L}_{23,N}(t;x,y).\end{aligned}\tag{4.8}$$

We first deal with the first term $\mathcal{E}^{\pm,L}_{21,N}$. Let $\sigma>k+1+\frac{1}{2}$, then

$$\begin{aligned}\big|\partial_\lambda E^{L,\pm}_{21}(2^Ns;x,y)\big|\lesssim&\sum_{k=0}^{1}\big\|V(\cdot)\langle\cdot\rangle^\sigma\big\|^2_{L^2}\big\|\partial_s^{k+1}R_V^\pm(2^{4N}s^4)\big\|_{L^2_\sigma\to L^2_{-\sigma}}\\ \lesssim&2^{-N}\lesssim 1,\quad k=0,1.\end{aligned}$$

Note that $s\in\operatorname{supp}\varphi_0\subset[\frac{1}{4},1]$, by using integration by parts again, we obtain that

$$\begin{aligned}\big|\mathcal{E}^{\pm,L}_{21,N}(t;x,y)\big|\lesssim&\frac{1}{1+|t|2^{4N}}\Big|\int_0^\infty e^{-it2^{4N}s^4}\partial_s\Big(s^{-5+2\alpha}\varphi_0(s)E^{L,\pm}_{21}(2^Ns;x,y)\Big)ds\Big|\\ \lesssim&\frac{1}{1+|t|2^{4N}}.\end{aligned}\tag{4.9}$$

Next, we deal with the second term $\mathcal{E}^{\pm,L}_{22,N}$. Let

$$F^\mp(p):=e^{\mp ip}F_0^\mp(p),\quad F_0^\mp(p)=\frac{1-e^{-p}e^{\pm ip}}{p},$$

then

$$\partial_sF^\pm(2^Ns|*-y|)=2^N|*-y|(F^\pm)'(2^Ns|*-y|):=e^{\pm i2^Ns|*-y|}s^{-1}F_1^\pm(2^Ns|*-y|),$$

where

$$F_1^\pm(p)=pe^{\mp ip}(F^\pm)'(p)=\frac{(\pm ip-1)+(p+1)e^{-p}e^{\mp ip}}{p}.$$

Thus

$$\begin{aligned}E^{L,\pm}_{22}(2^Ns;x,y)=&e^{\pm i2^Ns(|x|+|y|)}s^{-1}\Big\langle VR_V(2^Ns)V\big(e^{\pm i2^Ns(|*-y|-|y|)}F_1^\pm(2^Ns|*-y|)\big)(\cdot),\\ &e^{\mp i2^Ns(|x-\cdot|-|x|)}F_0^\mp(2^Ns|x-\cdot|)\Big\rangle:=e^{\pm i2^Ns(|x|+|y|)}s^{-1}\widetilde{E}^{L,\pm}_{22}(2^Ns;x,y).\end{aligned}$$

Furthermore,

$$\mathcal{E}^{\pm,L}_{22,N}(t;x,y)=\frac{2^{(2+\alpha)N}}{1+|t|2^{4N}}\int_0^\infty e^{-it2^{4N}s^4}e^{\pm i2^Ns(|x|+|y|)}s^{-3+\alpha}\varphi_0(s)\widetilde{E}^{L,\pm}_{22}(2^Ns;x,y)ds.$$

Note that

$$\begin{aligned}&\left|\partial_s^k\left(e^{\pm i2^N s(|*-y|-|y|)}F_1^{\pm}(2^N s|*-y|)\right)\right|\lesssim 2^{kN}\langle *\rangle, &&k=0,1,\\ &\left|\partial_s^k\left(e^{\mp i2^N s(|x-\cdot|-|x|)}F_0^{\mp}(2^N s|x-\cdot|)\right)\right|\lesssim 2^{kN}\langle \cdot\rangle, &&k=0,1.\end{aligned}$$

Since $|V(x)|\lesssim(1+|x|)^{-4-}$, then by Hölder's inequality, we have

$$\begin{aligned}\left|\partial_s\widetilde{E}_{22}^{L,\pm}(2^N s;x,y)\right|&\lesssim\sum_{k=0}^{1}2^{(1-k)N}\left\|V(\cdot)\langle\cdot\rangle^{\sigma+1-k}\right\|_{L^2}^2\left\|\partial_s^k R_V^{\pm}(2^{4N}s^4)\right\|_{L^2_\sigma\to L^2_{-\sigma}}\\ &\lesssim 2^{-2N}\lesssim 1.\end{aligned}$$

By Lemma 2.1 with $z=(x,y)$ and

$$\Psi(z)=|x|+|y|,\Phi(2^N s;z)=\widetilde{E}_{22,N}^{L,\pm}(2^N s;x,y),$$

we get that $\mathcal{E}_{22,N}^{\pm,L}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. By the same argument we get that $\mathcal{E}_{23,N}^{\pm,L}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. By (4.8), we have

$$|\mathcal{E}_{2,N}^{\pm,L}(t;x,y)|\lesssim\frac{1}{1+|t|2^{4N}}+(1+|t|2^{4N})\Theta_{N_0,N}(t)\lesssim(1+|t|2^{4N})\Theta_{N_0,N}(t).$$

Similarly, we obtain that $\mathcal{E}_{1,N}^{\pm,L}$ is bounded by $(1+|t|2^{4N})\Theta_{N_0,N}(t)$. By (4.7), we obtain that $L_{2,N}^{\pm}$ is bounded by $2^{(2+\alpha)N}\Theta_{N_0,N}(t)$. Thus $L_{2,N}^{\pm}$ is bounded by $2^{(3+\alpha)N}\Theta_{N_0,N}(t)$.

Finally, by the same summing way with the proof of (2.10), we immediately obtain the desired conclusion (4.6). □

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