Ann. Appl. Math.
doi: 10.4208/aam.OA-2025-0001

Vol. **41**, No. 1, pp. 77-111
February 2025

The H^p-H^q Estimates for a Class of Dispersive Equations with Finite Type Geometry

Qingquan Deng[1,*] and Xuejian Meng[2]

[1] *School of Mathematics and Statistics and Key Laboratory of Nonlinear Analysis & Applications (Ministry of Education), Central China Normal University, Wuhan, Hubei 430079, China*
[2] *School of Mathematics and Statistics, and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, Hubei 430079, China*

Received 1 January 2025; Accepted (in revised version) 23 February 2025

Dedicated to the celebration of the 70th birthday of Professor Avy Soffer

---

Abstract. This paper studies the H^p-H^q estimates of a class of oscillatory integrals related to dispersive equations

$$\begin{cases} i\partial_t u(t,x)=Q(D)u(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x)=u_0(x), & x\in\mathbb{R}^n, \end{cases}$$

under the assumption that the level hypersurfaces are convex and of finite type. As applications, we obtain the decay estimates for the solutions of higher order homogeneous and inhomogeneous Schrödinger equations.

AMS subject classifications: 42B30, 42B37

Key words: Dispersive equations, H^p-H^q estimates, finite type geometry, decay estimates.

---

*Corresponding author.
Emails: `dengq@ccnu.edu.cn` (Q. Deng), `mengxuejian@mails.ccnu.edu.cn` (X. Meng)

1 Introduction

1.1 Backgrounds

In this paper, we mainly study the H^p-H^q estimates of the solution for the following Cauchy problem of dispersive equation:

$$\begin{cases} i\partial_t u(t,x) = Q(D)u(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x)=u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $Q:\mathbb{R}\to\mathbb{R}$ is a phase function, $D=-i(\partial_{x_1},\cdots,\partial_{x_n})$ with $n\geq 2$ and H^p $(0<p<\infty)$ are Hardy space, The operator $Q(D)$ is defined by

$$Q(D)f=\mathscr{F}Q(\xi)\mathscr{F}^{-1}f,$$

where $\mathscr{F}$ denotes Fourier transform and $\mathscr{F}^{-1}$ is its inverse. For $u_0\in\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), the solution of (1.1) is given by

$$u(t,\cdot):=e^{-itQ(D)}u_0=\mathscr{F}^{-1}(e^{-itQ(\xi)}\hat{u}_0). \tag{1.2}$$

When $Q(\xi)=|\xi|^2$, it is well known that (1.1) represents free Schrödinger equation and the solution operator $e^{-it\Delta}$ satisfies with the following sharp $L^p-L^{p'}$ estimates

$$||e^{-it\Delta}||_{L^p-L^{p'}}\leq C|t|^{\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}, \tag{1.3}$$

where $t\neq 0$, $p\in[1,2]$ and $\frac{1}{p}+\frac{1}{p'}=1$ (see e.g., [28]). Notice that $H^p=L^p$ when $1<p<\infty$ and H^1 (resp. L^∞) is a subspace of L^1 (resp. BMO), then one can rewrite (1.3) as

$$||e^{-it\Delta}||_{H^p-H^{p'}}\leq C|t|^{\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}, \tag{1.4}$$

where $H^{p'}=BMO$ when $p=1$. Thus it is of interest to study the H^p-H^q estimates of $e^{itQ(D)}$ for $p<1$, which are natural extensions of the decay estimates for $p\geq 1$.

Generally speaking, one can study the H^p-H^q estimates for generalized propagator $e^{-itQ(D)}$ in terms of the estimates of fundamental solution $\mathscr{F}^{-1}(e^{itQ(\xi)})$ $(t\neq 0)$, which is depending on the geometry of the level set

$$\Sigma=\{\xi:\ |Q(\xi)|=1\}.$$

Here Σ is usually a compact connected smooth hypersurface of $\mathbb{R}^n$ with certain geometric assumptions. In particular, Miyachi [26] considered the singular multipliers

$$\psi(\xi)|\xi|^{-b}e^{i|\xi|^a},$$