

The H^p - H^q Estimates for a Class of Dispersive Equations with Finite Type Geometry

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Abstract. This paper studies the H^p-H^q estimates of a class of oscillatory integrals related to dispersive equations

$$\begin{cases} i\partial_t u(t, x) = Q(D)u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

under the assumption that the level hypersurfaces are convex and of finite type. As applications, we obtain the decay estimates for the solutions of higher order homogeneous and inhomogeneous Schrödinger equations.

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1 Introduction

1.1 Backgrounds

In this paper, we mainly study the H^p-H^q estimates of the solution for the following Cauchy problem of dispersive equation:

$$\begin{cases} i\partial_t u(t, x) = Q(D)u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.1)$$

Here $Q: \mathbb{R} \rightarrow \mathbb{R}$ is a phase function, $D = -i(\partial_{x_1}, \dots, \partial_{x_n})$ with $n \geq 2$ and H^p ($0 < p < \infty$) are Hardy space, The operator $Q(D)$ is defined by

$$Q(D)f = \mathcal{F}Q(\xi)\mathcal{F}^{-1}f,$$

where $\mathcal{F}$ denotes Fourier transform and $\mathcal{F}^{-1}$ is its inverse. For $u_0 \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), the solution of (1.1) is given by

$$u(t, \cdot) := e^{-itQ(D)}u_0 = \mathcal{F}^{-1}(e^{-itQ(\xi)}\hat{u}_0). \quad (1.2)$$

When $Q(\xi) = |\xi|^2$, it is well known that (1.1) represents free Schrödinger equation and the solution operator $e^{-it\Delta}$ satisfies with the following sharp $L^p-L^{p'}$ estimates

$$\|e^{-it\Delta}\|_{L^p-L^{p'}} \leq C|t|^{\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}, \quad (1.3)$$

where $t \neq 0$, $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (see e.g., [28]). Notice that $H^p = L^p$ when $1 < p < \infty$ and H^1 (resp. L^∞) is a subspace of L^1 (resp. BMO), then one can rewrite (1.3) as

$$\|e^{-it\Delta}\|_{H^p-H^{p'}} \leq C|t|^{\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}, \quad (1.4)$$

where $H^{p'} = BMO$ when $p = 1$. Thus it is of interest to study the H^p-H^q estimates of $e^{itQ(D)}$ for $p < 1$, which are natural extensions of the decay estimates for $p \geq 1$.

Generally speaking, one can study the H^p-H^q estimates for generalized propagator $e^{-itQ(D)}$ in terms of the estimates of fundamental solution $\mathcal{F}^{-1}(e^{itQ(\xi)})$ ($t \neq 0$), which is depending on the geometry of the level set

$$\Sigma = \{\xi: |Q(\xi)| = 1\}.$$

Here Σ is usually a compact connected smooth hypersurface of $\mathbb{R}^n$ with certain geometric assumptions. In particular, Miyachi [26] considered the singular multipliers

$$\psi(\xi)|\xi|^{-b}e^{i|\xi|^a},$$

where $\psi(\xi)$ is smooth and vanishes around origin, $a > 0$ and $b \in \mathbb{R}$, and established in [26, Theorem 4.1] the boundedness for multipliers between the Hardy spaces H^p , BMO and Lipschitz spaces Λ_s under certain restrictions on a and b . Notice that in this case $Q(\xi) = |\xi|^a$ and the hypersurface Σ is exactly the spherical surface $\mathbb{S}^{n-1}$, which has constant Gaussian curvature. The author also mentioned in [26, Remark 4.2] that the argument can also be applied to the multipliers

$$\psi(\xi)(e^{iQ_1(\xi)^{\alpha_1}}h_1(\xi) + e^{iQ_2(\xi)^{\alpha_2}}h_2(\xi)),$$

where h_i ($i=1,2$) are smooth homogeneous functions of degree $-b$ and Q_i ($i=1,2$) are smooth positive homogeneous functions of degree one such that

$$\Sigma_i = \{\xi : |Q_i(\xi)| = 1\}, \quad i=1,2,$$

have nonvanishing Gaussian curvature. Notice that the phase function both $|\xi|^a$ and $Q_i(\xi)^{\alpha_i}$ ($i=1,2$) are homogeneous. For the propagator operator $e^{itQ(D)}$ ($t \neq 0$) with Q satisfying the condition in [26] one can use scaling $\xi \rightarrow t^{\frac{1}{a}}\xi$ ($t \neq 0$) to get the optimal $H^p - H^q$ estimates. We remark that Σ has nonvanishing Gaussian curvature is equivalent to the so-called nondegenerate condition on Q , based on which the $H^p - H^q$ estimates with $1 \leq p \leq q$ of propagator $e^{itQ(D)}$ have also been extensively studied for inhomogeneous Q , (see e.g., [1-4, 11, 22, 23]). Among them all, we would like to emphasize the work of Cui [11], where the following decay estimates have been established

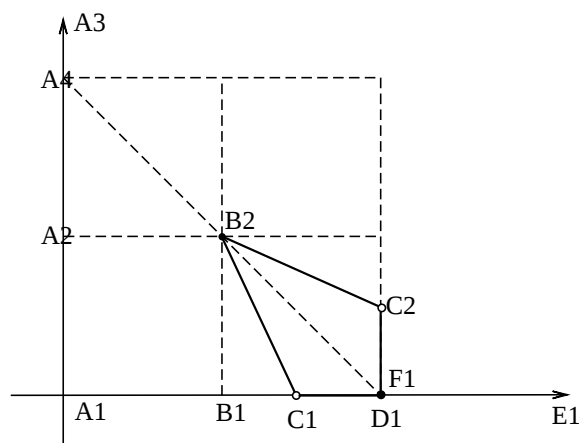
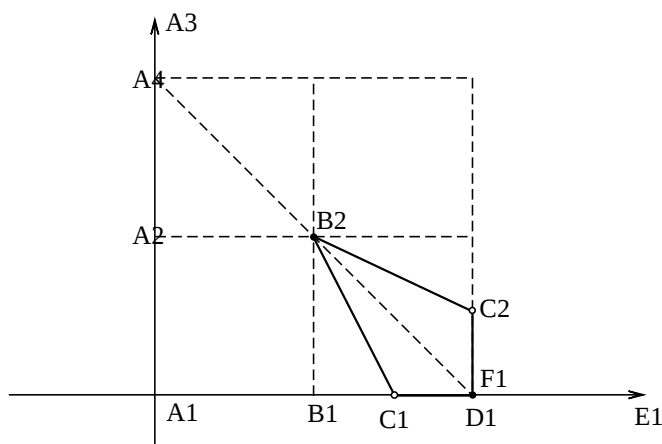
$$\|e^{itQ(D)}\|_{L^p-L^q} \leq C(T)|t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})}, \quad 0 < |t| < T, \quad (1.5)$$

where $T > 0$, Q is a inhomogeneous elliptic polynomial with the principal Q_m being nondegenerate and $(\frac{1}{p}, \frac{1}{q}) \in \square_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$, $\square_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$ is a closed quadrangle by the four vertex points

$$\tilde{A} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \tilde{B} = \left(1, \frac{1}{p_0}\right), \quad \tilde{C} = (1, 0), \quad \tilde{D} = \left(\frac{1}{p_0}, 0\right),$$

where $\frac{1}{p_0} + \frac{1}{p_0} = 1$ and $p_0 = \frac{2(m-1)}{m}$ with $m \geq 2$.

If the level set Σ has zero Gaussian curvature at some points of Σ , it would become more difficult to estimate the oscillatory integral $\mathcal{F}^{-1}(e^{itQ(\xi)})$ ($t \neq 0$) due to the failure of the principle of stationary phase (see e.g., [27]). In fact, there exist elliptic polynomials such that their level sets have zero Gaussian curvature at some points, for instance $\xi_1^m + \dots + \xi_n^m$ ($m=4, 6, \dots$) and $\xi_1^4 + 6\xi_1^2\xi_2^2 + \xi_2^4$. Motivated by these examples, based on [5, Theorem B], Zheng et al. [30] established the $L^p - L^q$ estimates by assuming Q is a homogeneous elliptic polynomial of order $m \geq 2$ and

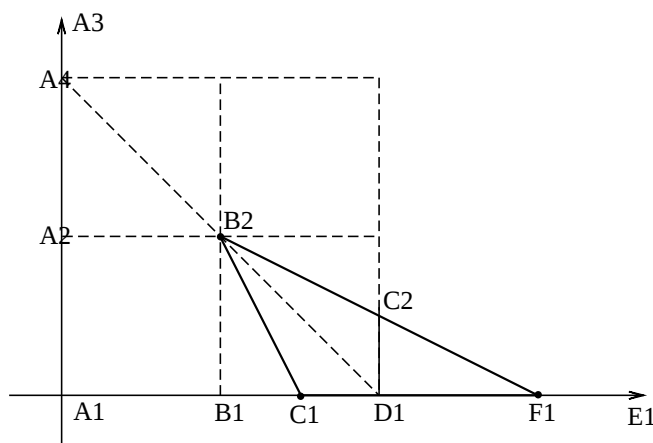
Figure 1: The $H^p - H^q$ estimates for $e^{itQ(D)}$ of Cui [11].Figure 2: The $L^p - L^q$ estimates for $e^{itQ(D)}$ of Zheng et al. [30].

Σ is a convex hypersurface of finite type $k \geq 2$ ($k \in \mathbb{N}$). Indeed, they proved that (see [30, Theorem 2.4])

$$\|e^{itQ(D)}\|_{L^p-L^q} \leq C|t|^{\frac{n}{m}(\frac{1}{q}-\frac{1}{p})}, \quad t \neq 0, \quad (1.6)$$

where $(\frac{1}{p}, \frac{1}{q}) \in \square_{A_1 B_1 C_1 D_1} \setminus \{B_1, D_1\}$ and $\square_{A_1 B_1 C_1 D_1}$ is a closed quadrangle by the four vertex points (see Fig. 2). Here $A_1 = (\frac{1}{2}, \frac{1}{2})$, $B_1 = (1, \frac{1}{\tau})$, $C_1 = (1, 0)$ and $D_1 = (\frac{1}{\tau'}, 0)$, where τ' is the conjugate index of τ with

$$\tau = \frac{2kn(m-1)}{m(2n+k-2)-2kn}. \quad (1.7)$$



Such result has been generalized to the inhomogeneous setting, see [12]. Notice that when $k=2$, $\square_{A_1B_1C_1D_1}$ is exactly the same as $\square_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$ in [11]. However the author did not mention whether the domain of $(\frac{1}{p}, \frac{1}{q})$ for L^p-L^q estimates is optimal as $k>2$. This optimality problem was partially answered in the work of Ding and Yao [12], where the authors obtained the H^p-H^q estimates

$$\|e^{itQ(D)}\|_{H^p-H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})}, \quad t \neq 0,$$

where Q satisfies the same assumptions as in [30] and $(\frac{1}{p}, \frac{1}{q}) \in \Delta_{A_2 B_2 D_2}$ (see Fig. 3). Here $A_2 = (\frac{1}{2}, \frac{1}{2})$, $B_2 = (\frac{1}{q_0}, 0)$ and $D_2 = (\frac{1}{p_0}, 0)$ with $2 \leq k \leq m$ and

$$\frac{1}{p_0} = \frac{m(2n+k-2)}{2kn}, \quad \frac{1}{q_0} = \frac{1}{2-p_0}.$$

When Q is a polynomial, Deng and Yao [14] considered the case of $Q(\xi) = P(\phi(\xi))$, where P is a real polynomial of order $m \geq 2$, $\phi(\xi)$ is a homogeneous smooth function, and the level set Σ is a smooth convex hypersurface, then the authors obtained the following attenuation estimates

$$\|e^{itP(\phi(D))}\|_{L^p_*-L^q_*} \leq \begin{cases} C|t|^{n|\frac{1}{q}-\frac{1}{p}|}, & 1 \leq |t| < T, \\ C|t|^{\frac{n}{m}(\frac{1}{q}-\frac{1}{p})}, & 0 < |t| < 1, \end{cases}$$

where $(\frac{1}{p}, \frac{1}{q}) \in \square_{A_3 B_3 C_3 D_3}$ is a closed quadrangle by the four vertex points $A_3 = (\frac{1}{2}, \frac{1}{2})$, $B_3 = (1, \frac{1}{\nu})$, $C_3 = (1, 0)$, $D_3 = (\frac{1}{\nu'}, 0)$ (see Fig. 4), ν' is the conjugate index of ν , $2 \leq k \leq$

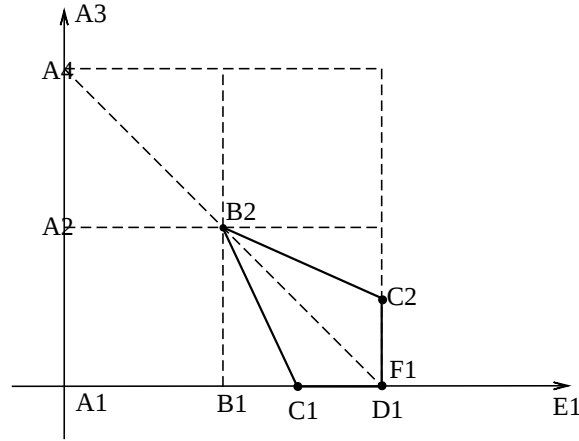


Figure 4: The $L_*^p - L_*^q$ estimates for $e^{itP(\phi(D))}$ of Deng and Yao [14].

$h(m, n, k)$

$$h(m, n, k) = \frac{m-2}{2(m-1)} + \frac{(m-k)(n-1)}{k(m-1)}, \quad \frac{1}{\nu} = \frac{h(m, n, k)}{n},$$

$$L_*^p - L_*^q = \begin{cases} L^1 - L^{\tau, \infty}, & \left(\frac{1}{p}, \frac{1}{q}\right) = \left(1, \frac{1}{\tau}\right), \\ L^{\tau', 1} - L^\infty, & \left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{1}{\tau'}, 0\right), \\ L^p - L^q, & \text{else.} \end{cases}$$

When Q is non-polynomial and satisfies **(H1)** and **(H2)**, Deng et al. [15] obtained the following $H^p - H^q$ decay estimate for $e^{itQ(|\nabla|)}$

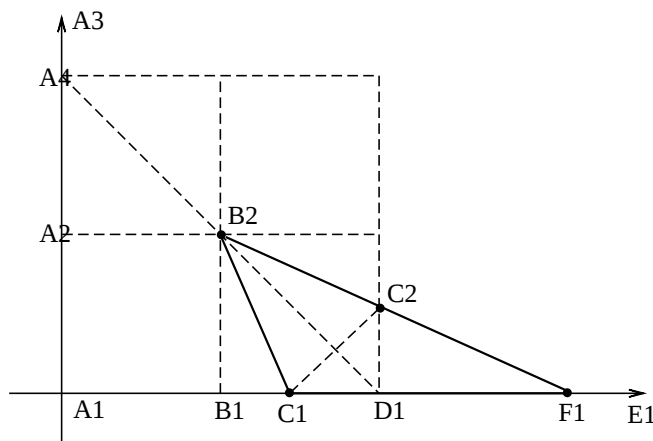
$$\|e^{itQ(|\nabla|)}\|_{H^p - H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p} - \frac{1}{q})},$$

where ∇ is the gradient, $m_1 \geq 2, m_2 > 0$, $m = \max\{m_1, m_2\}$, and $(\frac{1}{p}, \frac{1}{q}) \in \Delta_{A_4 B_4 D_4}$ is a triangle with vertices $A_4: (\frac{1}{2}, \frac{1}{2})$, $B_4: (\frac{m_1}{2}, 0)$, $D_4: (\frac{m_1}{2(m_1-1)}, 0)$ (see Fig. 5).

Now by restricting $\Delta_{A_2 B_2 D_2}$ with $p, q \geq 1$ we observed that the range for $L^p - L^q$ estimates of $e^{itQ(D)}$ is larger than $\square_{A_1 B_1 C_1 D_1}$ when $k > 2$. That is, by studying $H^p - H^q$ estimates of $e^{itQ(D)}$, one can actually improve the range for $L^p - L^q$ estimates, which also reveals that it is meaningful to consider the $H^p - H^q$ estimates of $e^{itQ(D)}$ for general Q .

1.2 Main result

Let $n \geq 2$ and $\phi(\xi)$ be a homogeneous function of degree one on $\mathbb{R}^n$ that is smooth and positive away from the origin. $\Sigma := \{\xi \in \mathbb{R}^n \mid \phi(\xi) = 1\}$ is the level set. Note that



$\nabla\phi(\xi)\neq 0$ for any $\xi\in\Sigma$ by homogeneity, hence Σ is a smooth compact hypersurface of $\mathbb{R}^n$. Let $k\in\mathbb{N}$ and $k\geq 2$, we say Σ is of finite type if there exist k and $C_k>0$ such that

$$\sum_{i=1}^k |\langle \eta, \nabla \rangle^j \phi(\xi)| \geq C_k, \quad \xi \in \Sigma, \quad \eta \in S^{n-1}, \quad (1.8)$$

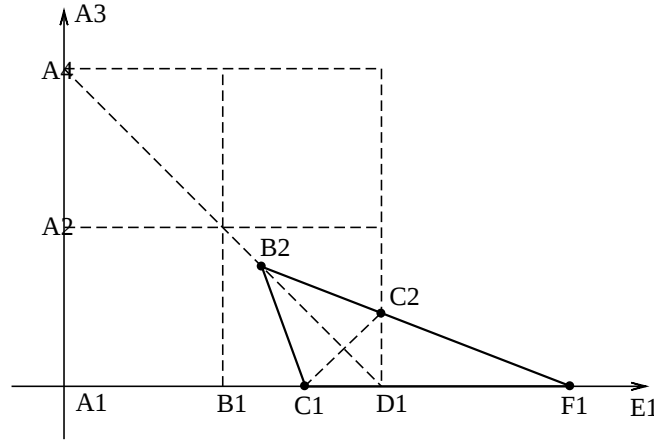
$$\Sigma \subset \{\eta \in \mathbb{R}^n \mid \langle \eta - \xi, \nabla \phi(\xi) \rangle \geq 0\} \quad \text{or} \quad \Sigma \subset \{\eta \in \mathbb{R}^n \mid \langle \eta - \xi, \nabla \phi(\xi) \rangle \leq 0\}.$$

(H1): There exists $m_1 > 0$, such that for any $\alpha > 2$, $\alpha \in \mathbb{N}$,

$$|P'(\xi)| \sim \xi^{m_1-1}, \quad |P''(\xi)| \sim \xi^{m_1-2}, \quad |P^{(\alpha)}(\xi)| \lesssim \xi^{m_1-\alpha}, \quad \xi \geq 1.$$

$$|P'(\xi)| \sim \xi^{m_2-1}, \quad |P''(\xi)| \sim \xi^{m_2-2}, \quad |P^{(\alpha)}(\xi)| \lesssim \xi^{m_2-\alpha}, \quad 0 < \xi < 1,$$

(H_Σ): $\Sigma = \{\xi \in \mathbb{R}^n \mid \phi(\xi) = 1\}$ is a smooth convex and of finite type, $k \geq 2$.

Figure 6: When $\gamma \geq 0$, $H^p - H^q$ estimate of $I^\gamma S(t)$.

Notice that these conditions were also introduced in references [9, 17], where the authors considered the dispersive estimates of $e^{itP(\phi(D))}$ and $e^{itP(|\nabla|)}$ respectively, and in [13], where the authors studied pointwise convergence of $e^{itP(|\nabla|)}$. Denote by $S(t) = e^{itP(\phi(D))}$ the linear propagator of Eq. (1.1) and $I^\gamma = \phi(D)^\gamma$ ($\gamma \in \mathbb{R}$), where the definition of $S(t)$ is given by (1.2) and

$$\phi(D)^\gamma f = \mathcal{F}^{-1}(\phi(\cdot))^\gamma \mathcal{F} f).$$

The main result is stated as follows.

Theorem 1.1. Assume that **(H1)** and **(H2)** are satisfied with $m_1 > 2$ and $m_2 > 0$. Let $m = \max\{m_1, m_2\}$, $\gamma \geq 0$ and $\frac{m_1}{k} - \frac{\gamma}{n} \geq 1$. Then for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_{m_1, \gamma}$ (see Fig. 6), we have

$$\|I^\gamma S(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n})} \|f\|_{H^p}. \quad (1.9)$$

Here $\mathfrak{J}_{m_1, \gamma}$ denotes the triangle with vertices

$$A_5: \left(\frac{1}{2} + \frac{k\gamma}{2n(m_1 - k)}, \frac{1}{2} - \frac{k\gamma}{2n(m_1 - k)}\right), \quad B_5: \left(\frac{m_1}{k} - \frac{\gamma}{n}, 0\right) \quad \text{and} \quad D_5: (\eta, 0)$$

with

$$\eta = 1 - \left(\frac{1}{2} - \frac{k\gamma}{2n(m_1 - k)}\right) \frac{\frac{m_1}{k} - \frac{\gamma}{n} - 1}{\frac{m_1}{k} - \frac{\gamma}{n} - \frac{1}{2} - \frac{k\gamma}{n(m_1 - k)}}. \quad (1.10)$$

The next result is concerning the estimates for $\gamma = 0$ in Theorem 1.1. We note that in this case we can choose $m_1 \geq 2$, which is slightly different from the assumption in Theorem 1.1.

Corollary 1.1. Assume that (H1) and (H2) are satisfied with $m_1 \geq \max\{2, k\}$ and $m_2 > 0$. Then for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_{m_1}$ (see Fig. 7), we have

$$\|S(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})} \|f\|_{H^p}. \quad (1.11)$$

Here $\mathfrak{J}_{m_1}$ denotes the triangle with vertices $A_6: (\frac{1}{2}, \frac{1}{2})$ and $B_6: (\frac{m_1}{k}, 0)$.

Remark 1.1. (i) Now by restricting $\Delta_{A_6 B_6 D_6}$ with $p, q \geq 1$ we observed that the range for $L^p - L^q$ estimates of $e^{itQ(D)}$ is larger than $\square_{A_3 B_3 C_3 D_3}$ when $k > 2$. For the non-homogeneous Q , we obtain the decay estimate of $H^p - H^q$ for all time $t \neq 0$, while Cui [11] and Deng-Yao [14] obtained the estimate of $L^p - L^q$ for any finite time $t \neq 0$.

(ii) When we take $\phi(\xi) = |\xi|$, this corresponds to the result in [15] for $k = 2$. Furthermore, the range of $\mathfrak{J}_{m_1, \gamma}$ is the same as that of $\Delta_{A_4 B_4 D_4}$, as well as the decay estimates are also the same.

The proof of main theorem is based on the phase space analysis. Notice that the lower frequency and the higher frequency enjoy different scalings, we make suitable decomposition in phase space and then Theorem 1.1 can be reduce to the estimates for different types of oscillatory integrals. Therefore, the standard stationary phase argument and the Van der corput lemma shall be involved. We note that since the symbols $P(\phi(\xi))$ can be inhomogeneous in Theorem 1.1, our method is quiet different from the work of Ding-Yao [12], where the authors used scaling to transfer the time-dependent multiplier $e^{itP(\xi)}$ to time independent one $e^{iP(\xi)}$. On the other hand, Cui [11] obtained the $L^p - L^q$ estimates locally in time for inhomogeneous symbols

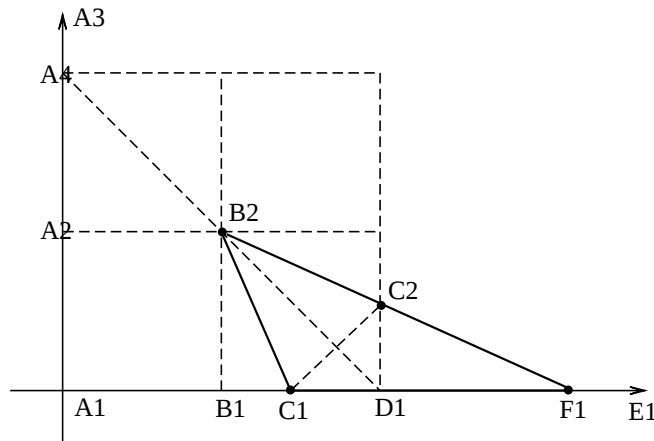


Figure 7: When $\gamma = 0$, $H^p - H^q$ estimate of $S(t)$.

(see (1.5)), in terms of the pointwise estimate for the kernel of $e^{itP(\xi)}$. However, the argument used in [11] can not be applied in our work, since we actually obtained the $H^p - H^q$ estimates globally in time for the propagator $S(t)$.

The paper is organized as follows. In Section 2, we introduce some notations and basic lemmas. Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.1. In Section 4, we apply Theorem 1.1 to specific linear Schrödinger equations with both homogeneous and inhomogeneous symbols.

2 Preliminaries

2.1 Some notations

Throughout this work, we write $X \lesssim Y$ to indicate there exists some constant $C > 0$ independent of X and Y such that $X \leq CY$. Let σ_1 be a C^∞ function satisfying

$$\sigma_1(\xi) \equiv 1 \quad \text{for } |\xi| \leq 1, \quad \text{supp } \sigma_1 \subset \{|\xi| \leq 2\} \quad \text{and} \quad 0 \leq \sigma_1 \leq 1.$$

We choose Ψ is of the same type of σ_1 and set

$$\sigma_2 = 1 - \sigma_1, \quad \Phi = 1 - \Psi.$$

Next, we introduce some function spaces. The $L^p(\mathbb{R}^n)$ based homogeneous Sobolev spaces $\dot{L}_p^s(\mathbb{R}^n)$ is the set of tempered distribution f satisfying $(-\Delta)^{\frac{s}{2}} f \in L^p(\mathbb{R}^n)$. We denote by $H^p(\mathbb{R}^n)$ ($0 < p \leq 1$) the real Hardy spaces and $\dot{\Lambda}^s(\mathbb{R}^n)$ the homogeneous Hölder space. We refer to the books [18, 24, 29] et al. for the definitions and their equivalent characterizations. One of the equivalent norms of the homogeneous Hölder space $\dot{\Lambda}^s(\mathbb{R}^n)$ will be given here. To do so, let Γ be a C^∞ function supported in $\{\xi: \frac{1}{2} \leq |\xi| \leq 2\}$ satisfying

$$\sum_{j \in \mathbb{Z}} \Gamma(2^{-j}|\xi|) = 1, \quad \xi \neq 0. \quad (2.1)$$

Denote by $\Gamma_j(\xi) = \Gamma(2^{-j}|\xi|)$ and $\Gamma_{j,\rho}(\xi) = \Gamma(2^{-j}\rho|\xi|)$ ($\rho > 0$), then it follows from the results of [18, 29] and [16] that for $s \geq 0$

$$\|f\|_{\dot{\Lambda}^s(\mathbb{R}^n)} \sim \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} |\mathcal{F}^{-1}(\Gamma_j \hat{f})(x)|.$$

Furthermore, by using the same argument of [10], we have an equivalent norm of $\dot{\Lambda}^s(\mathbb{R}^n)$ with an extra parameter. Precisely, for any $\rho \geq 0$

$$c \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} \rho^{-s} |\mathcal{F}^{-1}(\Gamma_{j,\rho} \hat{f})(x)| \leq \|f\|_{\dot{\Lambda}^s(\mathbb{R}^n)} \leq C \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} \rho^{-s} |\mathcal{F}^{-1}(\Gamma_{j,\rho} \hat{f})(x)|, \quad (2.2)$$

where $s \geq 0$ and the positive constants c and C are independent of ρ .

Since $\phi(\xi)$ is a positive smooth homogeneous function degree one that is bounded on sphere $\mathbb{S}^{n-1}$, there exist constant $0 < C_1 < 1$ and $C_2 > 1$ that satisfy

$$C_1|\xi| \leq \phi(\xi) \leq C_2|\xi|.$$

Let

$$\tilde{\Gamma}_k(\xi) = \Gamma(2^{-k}\phi(\xi)) \quad \text{and} \quad \tilde{\Gamma}_{k,\rho}(\xi) = \Gamma(2^{-k}\rho\phi(\xi)),$$

by simple calculation, we have $\tilde{\Gamma}_j(\xi)\Gamma_k(\xi) = 0$ for $|j-k| > C_3$, where

$$C_3 = \max \left\{ \left[\log_2 \frac{4}{C_1} \right] + 1, \left[\log_2 \frac{1}{4C_2} \right] + 1 \right\}.$$

Then one has

$$\begin{aligned} \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} |\mathcal{F}^{-1}(\Gamma_j \hat{f})(x)| &= \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} \left| \mathcal{F}^{-1} \left(\Gamma_j \sum_{k \in \mathbb{Z}} \tilde{\Gamma}_k(\xi) \tilde{\Gamma}_k(\xi) \hat{f} \right)(x) \right| \\ &\leq \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{ks+(j-k)s} \left| \mathcal{F}^{-1} \left(\Gamma_j \sum_{k=j-C_3}^{j+C_3} \tilde{\Gamma}_k \tilde{\Gamma}_k \hat{f} \right)(x) \right| \\ &\leq \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{C_3 s} \sum_{k=j-C_3}^{j+C_3} 2^{ks} |\mathcal{F}^{-1}(\tilde{\Gamma}_k \hat{f})(x)| \\ &\leq 2^{C_3 s} 2C_3 \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} |\mathcal{F}^{-1}(\tilde{\Gamma}_j \hat{f})(x)|. \end{aligned}$$

Similarly, we have

$$\sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} \rho^{-s} |\mathcal{F}^{-1}(\Gamma_{j,\rho} \hat{f})(x)| \lesssim \sup_{\substack{x \in \mathbb{R}^n \\ j \in \mathbb{Z}}} 2^{js} \rho^{-s} |\mathcal{F}^{-1}(\tilde{\Gamma}_{j,\rho} \hat{f})(x)|. \quad (2.3)$$

In the rest of the paper, we would use X instead of $X(\mathbb{R}^n)$ for simplicity when $X(\mathbb{R}^n)$ are certain function spaces.

2.2 Some lemmas

Before proving Theorem 1.1, we need some lemmas. The following one is related to the Fourier transform of a measure carried on a smooth hypersurface (see [5]).

Lemma 2.1. Suppose Σ is a compact smooth convex hyper-surface of finite type k in $\mathbb{R}^n$. For each $\eta \in S^{n-1}$, let $\xi_{\pm}$ be the two points of Σ whose outward normal directions are $\pm\eta$. If $\psi \in C^\infty(\Sigma)$ and define

$$\widehat{\psi d\sigma}(\lambda\eta) = \int_{\Sigma} e^{i\lambda\langle\eta, \xi\rangle} \psi(\xi) d\sigma(\xi),$$

then

$$\widehat{\psi d\sigma}(\lambda\eta) = e^{i\lambda\langle\eta, \xi_+\rangle} H_+(\lambda) + e^{i\lambda\langle\eta, \xi_-\rangle} H_-(\lambda) + H_\infty(\lambda), \quad \lambda > 0,$$

where $H_{\pm}, H_\infty \in C^\infty((0, \infty))$, and for every $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exist constants C_j and $C_{j,N}$ depending on Σ such that

$$|H_{\pm}^{(j)}(\lambda)| \leq C_j \lambda^{-j - \frac{n-1}{k}},$$

and

$$|H_\infty^{(j)}(\lambda)| \leq C_{j,N} \lambda^{-N} \quad \text{for } N \geq 0.$$

We also need the Van der Corput lemma, which is a basic tool to deal with oscillatory integrals, one can see for example [27] and [18].

Lemma 2.2. Suppose ϕ is real-valued and smooth in (a, b) satisfying $|\phi^k(x)| \geq 1$ for all $x \in (a, b)$. Then for any function ψ on (a, b) with an integrable derivative, there exist constant c_k independent of ϕ , ψ and λ such that

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-\frac{1}{k}} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right]$$

holds when:

- (i). $k \geq 2$, or
- (ii). $k = 1$ and $\phi'(x)$ is monotonic.

The theorem of Mihlin multiplier and the interpolation for function spaces will be needed, see [7] and [19], respectively.

Lemma 2.3. Let $0 < p \leq \infty$ and $l = [n|1/p - 1/2|] + 1$. If $a \in C^k(\mathbb{R}^n \setminus \{0\})$ and satisfies with

$$|D^\mu a(\xi)| \leq C_\mu |\xi|^{-|\mu|} \quad \text{for } |\mu| \leq l,$$

then $a \in \mathcal{M}(H^p, H^p)$.

Lemma 2.4. Assume that $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p_0, p_1, q_0, q_1 \leq \infty$ and either $p_0 + q_0 < \infty$ or $p_1 + q_1 < \infty$ and $0 < \theta < 1$. Let $s_0, s_1, p_0, q, p_1, q_1$ satisfy

$$\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

One has

$$[\dot{F}_{s_0}^{p_0, q_0}(\mathbb{R}^n), \dot{F}_{s_1}^{p_1, q_1}(\mathbb{R}^n)]_\theta = \dot{F}_s^{p, q}(\mathbb{R}^n),$$

where $\dot{F}_{s_0}^{p_0, q_0}$ denotes the homogeneous Triebel-Lizorkin space.

3 The proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Let the smooth functions σ_1, σ_2, Φ and Ψ be defined in Section 2, notice that $e^{iP(\phi(\xi))}$ is the symbol of linear propagator $S(t)$,

$$e^{itP((\phi(\xi)))} = \sigma_1(\phi(\xi))e^{itP(\phi(\xi))} + \sigma_2(\phi(\xi))e^{itP(\phi(\xi))}.$$

Accordingly, we have

$$S(t) = S_1(t) + S_2(t).$$

For $t > 0$, denote by

$$\Phi_t(\xi) = \Phi(t^{\frac{1}{m_2}}\xi) \quad \text{and} \quad \Psi_t(\xi) = \Psi(t^{\frac{1}{w_2}}\xi),$$

where m_2 is given in assumption **(H2)**, and then

$$S_1(t) = S_{11}(t) + S_{12}(t),$$

where $S_{11}(t)$ has symbol

$$\Phi_t(\phi(\xi))\sigma_1(\phi(\xi))e^{itP(\phi(\xi))}$$

and $S_{12}(t)$ has symbol

$$\Psi_t(\phi(\xi))\sigma_1(\phi(\xi))e^{itP(\phi(\xi))}.$$

It follows from Plancherel's identity that

$$\|I^\gamma S_{11}(t)f\|_{L^2} \lesssim \|\phi(D)^\gamma f\|_{L^2}, \quad \|I^\gamma S_{12}(t)f\|_{L^2} \lesssim \|\phi(D)^\gamma f\|_{L^2},$$

and similarly,

$$\|I^\gamma S_1(t)f\|_{L^2} \lesssim \|\phi(D)^\gamma f\|_{L^2}, \quad \|I^\gamma S_2(t)f\|_{L^2} \lesssim \|\phi(D)^\gamma f\|_{L^2}.$$

3.1 Decay estimates for low frequency

Lemma 3.1. *Let $\gamma \geq 0$ and $m_2 > 0$ be defined in assumption (H2). Then we have for $0 < p < 1$,*

$$\|I^\gamma S_{11}(t)f\|_{L^\infty} \lesssim (1+|t|)^{-\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n})} \|f\|_{H^p}, \quad (3.1)$$

and for $p=1$,

$$\|I^\gamma S_{11}(t)f\|_{L^\infty} \lesssim (1+|t|)^{-\frac{n}{m_2}(1+\frac{\gamma}{n})} \|f\|_{L^1}. \quad (3.2)$$

Proof. We only prove the inequality (3.1), since the proof for (3.2) shares exactly the same procedures. To this end, notice that

$$(H^p)^* = \dot{\Lambda}^\beta,$$

where $\beta = n(\frac{1}{p} - 1)$ and $\dot{\Lambda}^\beta$ is the Hölder space, we may assume that β is a positive integer for simplicity. Denote by $K_1(t, \cdot)$ the kernel of $I^\gamma S_{11}(t)$ and write

$$\Omega_{x,1}(t, y) = K_1(t, x - y).$$

For any $f \in H^p$,

$$\|I^\gamma S_{11}(t)f\|_{L^\infty} = \sup_x |\langle \Omega_{x,1}(t, \cdot), f \rangle| \leq \sup_x \|\Omega_{x,1}(t, \cdot)\|_{\dot{\Lambda}^\beta} \|f\|_{H^p}. \quad (3.3)$$

For any $x \in \mathbb{R}^n$,

$$\begin{aligned} \sup_x \|\Omega_{x,1}(t, \cdot)\|_{\dot{\Lambda}^\beta} &\lesssim \left| \int_{\mathbb{R}^n} e^{itP(\phi(\xi))} \Psi_t(\phi(\xi)) \sigma_1(\phi(\xi)) \phi(\xi)^\gamma |\xi|^\beta e^{i\langle x, \xi \rangle} d\xi \right| \\ &\leq \int_{\mathbb{R}^n} |\Psi_t(\phi(\xi)) \sigma_1(\phi(\xi))| |\phi(\xi)^\gamma| |\xi|^\beta d\xi \\ &= \int_{\mathbb{R}^n} |\Psi(\phi(\xi)) \sigma_1(t^{-\frac{1}{m_2}} \phi(\xi))| t^{-\frac{1}{m_2}} |\xi|^\beta |\phi(t^{-\frac{1}{m_2}} \xi)^\gamma| t^{-\frac{n}{m_2}} d\xi \\ &= |t|^{-\frac{n}{m_2}(\frac{1}{p}-1)} |t|^{-\frac{\gamma}{m_2}} |t|^{-\frac{n}{m_2}} \int_{|\phi(\xi)| \leq 2} |\Psi(\phi(\xi)) \sigma_1(t^{-\frac{1}{m_2}} \phi(\xi))| |\phi(\xi)^\gamma| |\xi|^\beta d\xi \\ &\lesssim |t|^{-\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n})}, \end{aligned} \quad (3.4)$$

which combined with (3.3) lead to (3.1) for $t > 1$. Moreover, since σ_1 is compactly supported it is easy to see from (3.4) that $\sup_x \|\Omega_{x,1}(t, \cdot)\|_{\dot{\Lambda}^\beta}$ is uniformly bounded in t . Thus we finish the proof of (3.1). $\square$

Lemma 3.2. *Let $\gamma \geq 0$ and $m_2 > 0$ be defined in assumption (H2). Then we have for $0 < p < 1$,*

$$\|I^\gamma S_{12}(t)f\|_{L^\infty} \lesssim (1+|t|)^{\max\{-\frac{n}{k}, -\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n})\}} \|f\|_{H^p}, \quad (3.5)$$

and for $p=1$,

$$\|I^\gamma S_{12}(t)f\|_{L^\infty} \lesssim (1+|t|)^{\max\{-\frac{n}{k}, -\frac{n}{m_2}(1+\frac{\gamma}{n})\}} \|f\|_{L^1}.$$

Proof. Similarly to the proof of Lemma 3.1, we only need to prove (3.5) for $t > 0$. The proof will be split into short time $0 < t \leq 4^{m_2}$ and large time $t > 4^{m_2}$.

For $0 < t \leq 4^{m_2}$, the same argument used in the proof of Lemma 3.1 can be applied. Notice that $(H^p)^* = \dot{\Lambda}^\beta$ with $\beta = n(\frac{1}{p}-1)$, we may also assume that β is a positive integer. Thus

$$\|I^\gamma S_{12}(t)f\|_{L^\infty} = \sup_x |\langle \Omega_{x,2}(t, \cdot), f \rangle| \leq \sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta} \|f\|_{H^p}, \quad (3.6)$$

where $\Omega_{x,2}(t, y) = K_2(t, x-y)$ with $K_2(t, \cdot)$ being the kernel of $I^\gamma S_{12}(t)$. By changing of variable, we obtain

$$\begin{aligned} \sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta} &\lesssim \sup_x \left| \int_{\mathbb{R}^n} e^{itP(\phi(\xi))} \Phi_t(\phi(\xi)) \sigma_1(\phi(\xi)) \phi(\xi)^\gamma |\xi|^\beta e^{i\langle x, \xi \rangle} d\xi \right| \\ &\leq \int_{\mathbb{R}^n} |\Phi_t(\phi(\xi)) \sigma_1(\phi(\xi)) \phi(\xi)^\gamma |\xi|^\beta| d\xi \\ &= \int_{\mathbb{R}^n} \left| \Phi(\phi(\xi)) \sigma_1(t^{-\frac{1}{m_2}} \phi(\xi)) |t^{-\frac{1}{m_2}} \xi|^\beta \phi(t^{-\frac{1}{m_2}} \xi)^\gamma t^{-\frac{n}{m_2}} \right| d\xi \\ &= |t|^{-\frac{\beta}{m_2}} |t|^{-\frac{\gamma}{m_2}} |t|^{-\frac{n}{m_2}} \int_{\{\xi: 1 < \phi(\xi) < 2^{\frac{2m_1}{m_2}+1}\}} |\Phi(\xi) \sigma_1(t^{-\frac{1}{m_2}} \xi)| \phi(\xi)^\gamma |\xi|^\beta d\xi \\ &\lesssim |t|^{-\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n})}, \end{aligned}$$

which combined with (3.6) imply (3.5) for $0 < t \leq 4^{m_2}$. We note that $\sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta}$ is also uniformly bounded in t . We hence finish the proof for (3.5) with $0 < t \leq 4^{m_2}$.

It remains to prove (3.5) for $t > 4^{m_2}$. By using the equivalent norm of $\dot{\Lambda}^\beta$ (see (2.3)), we have

$$\begin{aligned} &\sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta} \\ &\lesssim \sup_{\substack{j \in \mathbb{Z} \\ x \in \mathbb{R}^n}} \left(\frac{2^j}{t^{1/m_2}} \right)^\beta \left| \int_{\mathbb{R}^n} e^{itP(\phi(\xi))} \Gamma(t^{\frac{1}{m_2}} 2^{-j} \phi(\xi)) \Phi_t(\phi(\xi)) \sigma_1(\phi(\xi)) \phi(\xi)^\gamma e^{i\langle x, \xi \rangle} d\xi \right| \end{aligned}$$

uniformly in $t > 0$. Then by changing of variable, it follows

$$\begin{aligned} & \sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta} \\ & \lesssim \sup_x t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in \mathbb{Z}} 2^{j\beta} \left| \int_{\mathbb{R}^n} e^{itP(t^{-\frac{1}{m_2}}\phi(\xi))} \Gamma(2^{-j}\phi(\xi)) \Phi(\phi(\xi)) \sigma_1(t^{-\frac{1}{m_2}}\phi(\xi)) \phi(\xi)^\gamma e^{i\langle x, \xi \rangle t^{-\frac{1}{m_2}}} d\xi \right| \\ & = \sup_x t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in \mathbb{Z}} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right|, \end{aligned} \quad (3.7)$$

where the phase function $\varphi(\xi, t, x)$ and $H(\xi)$ are defined by

$$\varphi(\xi, t, x) = tP(2^j t^{-\frac{1}{m_2}} \phi(\xi)) + \langle x, \xi \rangle 2^j t^{-\frac{1}{m_2}},$$

and

$$H(\xi) = \Gamma(\phi(\xi)) \Phi(2^j \phi(\xi)) \sigma_1(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \phi(\xi)^\gamma, \quad (3.8)$$

respectively. Note that $\Phi(\phi(\xi)) = 0$ if $\phi(\xi) < 1$ and σ_1 is compactly supported, we obtain

$$\sup_x \|\Omega_{x,2}\|_{\dot{\Lambda}^\beta} \lesssim \sup_x t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j=0}^{\log_2 2t^{\frac{1}{m_2}}} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right|. \quad (3.9)$$

For any fixed $|x| \neq 0$, we define sets

$$\begin{aligned} E_1 &= \{0 \leq j \leq \log_2 2t^{\frac{1}{m_2}} : 2^{j(m_2-1)-10} \geq t^{-\frac{1}{m_2}} |x|\}, \\ E_2 &= \{0 \leq j \leq \log_2 2t^{\frac{1}{m_2}} : M\alpha n 2^{j(m_2-1)+10} \leq t^{-\frac{1}{m_2}} |x|\}, \end{aligned}$$

and

$$E_3 = \{0 \leq j \leq \log_2 2t^{\frac{1}{m_2}} : 2^{j(m_2-1)-10} < t^{-\frac{1}{m_2}} |x| < M\alpha n 2^{j(m_2-1)+10}\},$$

where M is a sufficiently large number. Then we have

$$\{j : 0 \leq j \leq \log_2 2t^{\frac{1}{m_2}}\} = E_1 \cup E_2 \cup E_3. \quad (3.10)$$

If $j \in E_1$, let $\xi = r\xi'$ and $\xi' \in \Sigma$, we use polar coordinates to write

$$\begin{aligned} & 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\ & = 2^{j(n+\beta+\gamma)} \left| \int_{\Sigma} \int_0^\infty G(r) \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r) e^{iP_1(r)} dr |\nabla \phi(\xi')|^{-1} d\sigma(\xi') \right| \\ & \leq 2^{j(n+\beta+\gamma)} \left| \int_{\Sigma} \int_0^\infty G(r) \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r) e^{iP_1(r)} dr \right| |\nabla \phi(\xi')|^{-1} d\sigma(\xi'), \end{aligned} \quad (3.11)$$

where $d\sigma(\xi')$ is the Lebesgue measure induced surface on Σ ,

$$P_1(r) = tP(2^j t^{-\frac{1}{m_2}} r) + 2^j t^{-\frac{1}{m_2}} r \langle x, \xi' \rangle,$$

and

$$G(r) = \Gamma(r) r^{n+\gamma-1} \quad (3.12)$$

is a C^∞ function supported in the interval $[\frac{1}{2}, 2]$. Noting that $0 \leq j \leq \log_2 2t^{\frac{1}{m_2}}$ and $r \in [\frac{1}{2}, 2]$, we obtain

$$2^j t^{-\frac{1}{m_2}} r \leq 4.$$

Thus if $j \in E_1$, one has

$$\begin{aligned} |P'_1(r)| &= |t(2^j t^{-\frac{1}{m_2}}) P'(2^j t^{-\frac{1}{m_2}} r) + 2^j t^{-\frac{1}{m_2}} \langle x, \xi' \rangle| \\ &\geq |t(2^j t^{-\frac{1}{m_2}}) P'(2^j t^{-\frac{1}{m_2}} r)| - 2^j t^{-\frac{1}{m_2}} |x| \\ &\gtrsim t(2^j t^{-\frac{1}{m_2}})^{m_2} - 2^j t^{-\frac{1}{m_2}} |x| \\ &\gtrsim 2^{jm_2}. \end{aligned}$$

Moreover,

$$\begin{aligned} |P''_1(r)| &= |t(2^j t^{-\frac{1}{m_2}})^2 P''(2^j t^{-\frac{1}{m_2}} r)| \\ &\leq t(2^j t^{-\frac{1}{m_2}})^2 (2^j t^{-\frac{1}{m_2}} r)^{m_2-2} \\ &\leq 2^{jm_2}, \end{aligned}$$

and similarly,

$$|P_1^{(k)}(r)| \lesssim 2^{jm_2}, \quad k \geq 3.$$

Then for any integer $m \geq 0$ and $r \in [\frac{1}{2}, 2]$, we have

$$\begin{aligned} \left| \partial_r^m \frac{1}{P_1(r)} \right| &= \left| \frac{\sum_\ell C_\ell P'_1(r)^{\alpha_\ell^1} P''_1(r)^{\alpha_\ell^2} \dots P_1^{(m+1)}(r)^{\alpha_\ell^{m+1}}}{P'_1(r)^{m+1}} \right| \\ &\lesssim 2^{-jm_2}, \end{aligned} \quad (3.13)$$

where $\alpha_\ell^1 + \alpha_\ell^2 + \dots + \alpha_\ell^{m+1} = m$, $\alpha_\ell^1 + 2\alpha_\ell^2 + \dots + (m+1)\alpha_\ell^{m+1} = 2m$ and C_ℓ is constant for each ℓ .

Denote by

$$F(r) = G(r) \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r).$$

Notice that r lies in the support of Γ and $0 \leq j \leq \log_2 2t^{\frac{1}{m_2}}$. It follows that

$$\partial_r \Phi(2^j r) = 2^j \Phi'(2^j r) = 0, \quad j \geq 3,$$

and

$$|\partial_r^\ell \sigma_1(2^j t^{-\frac{1}{m_2}} r)| = |(2^j t^{-\frac{1}{m_2}})^\ell| |\sigma_1^{(\ell)}(2^j t^{-\frac{1}{m_2}} r)| \leq |2^j t^{-\frac{1}{m_2}}|^\ell \lesssim 1, \quad \ell \in \mathbb{N}_+.$$

Thus thus for arbitrary integer $N \geq 0$

$$\begin{aligned} |\partial_r^N F(r)| &= \left| \sum_{\ell=0}^N C_N^\ell \partial_r^{N-\ell} (\Gamma(r) r^{n+\gamma-1}) \partial_r^\ell (\Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m}} r)) \right| \\ &\leq \sum_{\ell=0}^N C_N^\ell |\partial_r^{N-\ell} (\Gamma(r) r^{n+\gamma-1})| \left(\sum_{m=0}^\ell C_\ell^m |\partial_r^{\ell-m} \Phi(2^j r)| |\partial_r^m \sigma_1(2^j t^{-\frac{1}{m}} r)| \right) \\ &\lesssim \sum_{\ell=0}^N C_N^\ell |\partial_r^{N-\ell} (\Gamma(r) r^{n+\gamma-1})| \lesssim 1. \end{aligned} \quad (3.14)$$

Now by integration by parts, we have for any $N \in \mathbb{N}_+$,

$$\begin{aligned} &\left| \int_0^\infty G(r) \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r) e^{iP_1(r)} dr \right| \\ &= \left| \int_0^\infty e^{iP_1(r)} F(r) dr \right| \\ &= \left| \int_0^\infty e^{iP_1(r)} D_r^N (F(r)) dr \right|, \end{aligned} \quad (3.15)$$

where D_r is an operator defined by

$$D_r F = \partial_r \left(\frac{1}{P_1'(r)} F \right) \quad \text{and} \quad D_r^N F = D_r \left(\frac{1}{P_1'(r)} D_r^{N-1} F \right). \quad (3.16)$$

It follows from (3.13) and (3.14) that

$$\begin{aligned} |D_r^N F| &= \left| \left(-\frac{1}{i} \right)^N \sum_{m=0}^N \sum_{l_1, \dots, l_N \in \Lambda_m^N} C_{m,N} \prod_{i=1}^N \partial_r^{l_i} \left(\frac{1}{P_1'(r)} \right) \partial_r^{N-m} F(r) \right| \\ &\lesssim 2^{-jm_2 N}, \end{aligned} \quad (3.17)$$

where $\Lambda_m^N = \{l_1, \dots, l_N \in \mathbb{Z}^+ : 0 \leq l_1 < \dots < l_N \leq N, l_1 + \dots + l_N = m\}$ and N enough large. Combined with (3.15) and (3.17), we obtain

$$\left| \int_0^\infty G(r) \Phi(2^j r) \sigma(2^j t^{-\frac{1}{m_2}} r) e^{iP(r)} dr \right| \lesssim 2^{-jm_2 N},$$

which combined with (3.11) implies

$$\begin{aligned}
& t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_1} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_1} 2^{j(n+\beta+\gamma)} 2^{-jm_2 N} \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} = t^{-\frac{n}{m_2} \left(\frac{1}{p} + \frac{\gamma}{n} \right)}. \tag{3.18}
\end{aligned}$$

Let us turn to the case $j \in E_2$. For any fixed $|x| \neq 0$, without loss of generality, we assume $|x_1| \geq \frac{|x|}{n}$. Noting that ϕ is a smooth positive homogeneous function and $\phi(\xi) \in \text{supp} \Gamma$, it follows $|\partial_{\xi_1}^\beta \phi(\xi)| \lesssim 1$ ($\beta \geq 0$). Since M is a large constant, we have for $j \in E_2$,

$$\begin{aligned}
|\partial_{\xi_1} \varphi(\xi, t, x)| &= |t 2^j t^{-\frac{1}{m_2}} P'(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \partial_{\xi_1} \phi(\xi) + x_1 2^j t^{-\frac{1}{m_2}}| \\
&\geq |x_1| 2^j t^{-\frac{1}{m_2}} - |t 2^j t^{-\frac{1}{m_2}} P'(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \partial_{\xi_1} \phi(\xi)| \\
&\geq \frac{|x|}{n} 2^j t^{-\frac{1}{m_2}} - 2^{jm_2} \gtrsim 2^{jm_2}. \tag{3.19}
\end{aligned}$$

Moreover,

$$\begin{aligned}
& |\partial_{\xi_1}^2 \varphi(\xi, t, x)| = |\partial_{\xi_1} (t 2^j t^{-\frac{1}{m_2}} P'(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \partial_{\xi_1} \phi(\xi) + x_1 2^j t^{-\frac{1}{m_2}})| \\
&= |t^{-\frac{1}{m_2}} P'(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \partial_{\xi_1}^2 \phi(\xi) + t (2^j t^{-\frac{1}{m_2}})^2 P''(2^j t^{-\frac{1}{m_2}} \phi(\xi)) (\partial_{\xi_1} \phi(\xi))^2| \\
&\lesssim |t 2^j t^{-\frac{1}{m_2}} (2^j t^{-\frac{1}{m_2}} \phi(\xi))^{m_2-1} \partial_{\xi_1}^2 \phi(\xi) + t (2^j t^{-\frac{1}{m_2}})^2 (2^j t^{-\frac{1}{m_2}} \phi(\xi))^{m_2-2} (\partial_{\xi_1} \phi(\xi))^2| \\
&\lesssim 2^{jm_2}, \tag{3.20}
\end{aligned}$$

and similarly,

$$|\partial_{\xi_1}^\alpha \varphi(\xi, t, x)| \lesssim 2^{jm_2}, \quad \alpha \geq 3.$$

Then similarly to (3.13), we have for arbitrary integer $m \geq 0$ and $\phi(\xi) \in [\frac{1}{2}, 1]$

$$\begin{aligned}
\frac{1}{\varphi'_{\xi_1}(\xi)} &= \left| \frac{\sum_{\ell} C_{\ell} \varphi'_{\xi_1}(\xi)^{\alpha_{\ell}^1} \varphi''_{\xi_1}(\xi)^{\alpha_{\ell}^2} \dots \varphi_{\xi_1}^{(m+1)}(\xi)^{\alpha_{\ell}^{m+1}}}{\varphi'_{\xi_1}(\xi)^{m+1}} \right| \\
&\lesssim 2^{-jm_2}, \tag{3.21}
\end{aligned}$$

where $\alpha_{\ell}^1 + \alpha_{\ell}^2 + \dots + \alpha_{\ell}^{m+1} = m$, $\alpha_{\ell}^1 + 2\alpha_{\ell}^2 + \dots + (m+1)\alpha_{\ell}^{m+1} = 2m$ and C_{ℓ} is constant for each ℓ . Similarly to (3.14), we have for any $k \in \mathbb{N}$,

$$\partial_{\xi_1} \Phi(2^j \phi(\xi)) = 0 \quad \text{with } j \geq 3, \quad |\partial_{\xi_1}^k \sigma_1(2^j t^{-\frac{1}{m_2}} \phi(\xi))| \lesssim 1,$$

and

$$\begin{aligned} |\partial_{\xi_1}^N H(\xi)| &= \left| \sum_{\ell=0}^N C_N^\ell \partial_{\xi_1}^{N-\ell} (\Gamma(\phi(\xi)) \phi(\xi)^\gamma) \partial_{\xi_1}^\ell (\Phi(2^j \phi(\xi)) \sigma_1(2^j t^{-\frac{1}{m_2}} \phi(\xi))) \right| \\ &\lesssim 1, \end{aligned} \quad (3.22)$$

where H is defined by (3.8). Then by integration by parts, for each integral in (3.7), we have

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\varphi(\xi, t, x)} H(\xi) d\xi &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{i\varphi(\xi, t, x)} H(\xi) d\xi_1 d\xi_2 \cdots d\xi_n \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{i\varphi(\xi, t, x)} D_{\xi_1}^N H(\xi) d\xi_1 d\xi_2 \cdots d\xi_n, \end{aligned} \quad (3.23)$$

where

$$D_{\xi_1} H = \partial_{\xi_1} \left(\frac{1}{i\varphi'_{\xi_1}} H \right) \quad \text{and} \quad D_{\xi_1}^N H = D_{\xi_1} \left(\frac{1}{i\varphi'_{\xi_1}} D_{\xi_1}^{N-1} H \right).$$

Notice that by (3.21) and (3.22), one has

$$\begin{aligned} |D_{\xi_1}^N H| &= \left| \left(-\frac{1}{i} \right)^N \sum_{m=0}^N \sum_{l_1, \dots, l_N \in \Lambda_m^N} C_{m,N} \prod_{i=1}^N \partial_{\xi_1}^{l_i} \frac{1}{\varphi'_{\xi_1}(\xi)} \partial_{\xi_1}^{N-m} H(\xi) \right| \\ &\lesssim 2^{-jm_2 N}, \end{aligned} \quad (3.24)$$

where $\Lambda_m^N = \{l_1, \dots, l_N \in \mathbb{Z}^+ : 0 \leq l_1 < \dots < l_N \leq N, l_1 + \dots + l_N = m\}$ and $N \in \mathbb{N}_+$. Combined with (3.23) and (3.24), we have

$$\left| \int_{\mathbb{R}} e^{i\varphi(\xi, t, x)} \Gamma(\phi(\xi)) \Phi(2^j \phi(\xi)) \sigma_1(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \phi(\xi)^\gamma d\xi_1 \right| \lesssim 2^{-jm_2 N}$$

and

$$\begin{aligned} &t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_2} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\ &\lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_2} 2^{j(n+\beta+\gamma)} (2^{jm_2})^{-N} \\ &\lesssim t^{-\frac{n+\beta+\gamma}{m_2}} = t^{-\frac{n}{m_2} (\frac{1}{p} + \frac{\gamma}{n})}. \end{aligned} \quad (3.25)$$

We finally consider the estimate for $j \in E_3$. Notice that

$$\frac{1}{m_2-1} \left(\log_2 \frac{|x|}{M\alpha n} t^{-\frac{1}{m_2}} - 10 \right) \leq j \leq \frac{1}{m_2-1} (\log_2 t^{-\frac{1}{m_2}} |x| + 10),$$

which means that E_3 has finite elements, that is $|E_3| \leq K_1$ with K_1 depending on m_2, M . For each $j \in E_3$, we write each integral in (3.9) in terms of polar coordinates,

$$\begin{aligned}\mathcal{R}_j(t, s) &:= \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} \Gamma(\phi(\xi)) \Phi(2^j \phi(\xi)) \sigma_1(2^j t^{-\frac{1}{m_2}} \phi(\xi)) \phi(\xi)^\gamma d\xi \\ &= \int_0^\infty G_1(r) e^{itP(2^j t^{-\frac{1}{m_2}} r)} \left(\int_\Sigma e^{i2^j t^{-\frac{1}{m_2}} r s \langle \omega, \xi' \rangle} \frac{d\sigma(\xi')}{|\nabla \phi(\xi')|} \right) dr,\end{aligned}$$

where

$$G_1(r) = \Gamma(r) r^{n+\gamma-1} \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r).$$

Since Σ is a compact smooth convex hypersurface in $\mathbb{R}^n$, the Gaussian map given by $\Pi: \xi' \in \Sigma \mapsto \frac{\nabla \phi(\xi')}{|\nabla \phi(\xi')|} \in \mathbb{S}^{n-1}$ is a homeomorphism from Σ to $\mathbb{S}^{n-1}$. Thus for given $\omega \in \mathbb{S}^{n-1}$, there exist $\xi'_\pm \in \Sigma$ such that $\pm\omega$ is the outward unit normal direction and by Euler's homogeneous formula

$$\langle \omega, \xi'_\pm \rangle = \left\langle \pm \frac{\nabla \phi(\xi'_\pm)}{|\nabla \phi(\xi'_\pm)|}, \xi'_\pm \right\rangle = \frac{\pm 1}{|\nabla \phi(\xi'_\pm)|}.$$

Hence let $s = |x|$ and $x = s\omega$, by Lemma 2.1, we have

$$\int_\Sigma \frac{e^{i\lambda \langle \omega, \xi' \rangle}}{|\nabla \phi(\xi')|} d\sigma(\xi') = e^{i\lambda \langle \omega, \xi'_+ \rangle} H_+(\lambda) + e^{i\lambda \langle \omega, \xi'_- \rangle} H_-(\lambda) + H_\infty(\lambda), \quad \lambda = 2^j t^{-\frac{1}{m_2}} r s, \quad (3.26)$$

where $H_\pm \in C^\infty((0, \infty))$, and there exist constants C_j depending on Σ such that

$$|H_\pm^{(j)}(\lambda)| \leq C_j \lambda^{-j-(n-1)/k} \quad \text{for } j \in \mathbb{N}_+, \quad (3.27)$$

and

$$|H_\infty^{(j)}(\lambda)| \leq C_j \lambda^{-N} \quad \text{for } j \in \mathbb{N}_+.$$

If follows from (3.26) that

$$\begin{aligned}\mathcal{R}_j(t, s) &= \int_0^\infty G_1(r) e^{itP(2^j t^{-\frac{1}{m_2}} r) + i2^j t^{-\frac{1}{m_2}} r s \langle \omega, \xi'_+ \rangle} H_+(2^j t^{-\frac{1}{m_2}} r s) dr \\ &\quad + \int_0^\infty G_1(r) e^{itP(2^j t^{-\frac{1}{m_2}} r) + i2^j t^{-\frac{1}{m_2}} r s \langle \omega, \xi'_- \rangle} H_-(2^j t^{-\frac{1}{m_2}} r s) dr \\ &\quad + \int_0^\infty G_1(r) e^{itP(2^j t^{-\frac{1}{m_2}} r)} H_\infty(2^j t^{-\frac{1}{m_2}} r s) dr \\ &= \mathcal{R}_{j1}(t, s) + \mathcal{R}_{j2}(t, s) + \mathcal{R}_{j3}(t, s).\end{aligned} \quad (3.28)$$

To estimate $\mathcal{R}_{j1}(t, s)$, we denote by

$$\begin{aligned} U_{j1}(t, r, s) &= tP(2^j t^{-\frac{1}{m_2}} r) + 2^j t^{-\frac{1}{m_2}} r s \langle \omega, \xi'_+ \rangle, \\ V_{j1}(t, r, s) &= \Gamma_0(r) r^{n+\gamma-1} \Phi(2^j r) \sigma_1(2^j t^{-\frac{1}{m_2}} r) H_+(2^j t^{-\frac{1}{m_2}} r s). \end{aligned}$$

It is easy to see from assumption **(H2)** that

$$|\partial_r^2 U_{j1}(t, r, s)| = |t(2^j t^{-\frac{1}{m_2}} r)^2 P''(2^j t^{-\frac{1}{m_2}} r)| \gtrsim 2^{jm_2}.$$

Since $j \in E_3$, it follows from (3.27) that for $N \geq 0$

$$|\partial_r^N V_{j1}(t, r, s)| \lesssim (2^j t^{-\frac{1}{m_2}} s)^{-\frac{n-1}{k}}. \quad (3.29)$$

Thus for $j \in E_3$, by Lemma 2.2, we obtain

$$\begin{aligned} |\mathcal{R}_{j1}(t, s)| &\lesssim (2^{jm_2})^{-\frac{1}{2}} (\|V_{j1}\|_\infty + \|\partial_r V_{j1}\|_{L^1}) \\ &\lesssim (2^{jm_2})^{-\frac{1}{k}} (2^j t^{-\frac{1}{m_2}} s)^{-\frac{n-1}{k}} \\ &\lesssim (2^{jm_2})^{-\frac{1}{k}} (2^j t^{-\frac{1}{m_2}} t^{\frac{1}{m_2}} 2^{j(m_2-1)-10})^{-\frac{n-1}{k}} \\ &\lesssim (2^{jm_2})^{-\frac{n}{k}}. \end{aligned}$$

The same procedures as above can be applied to obtain

$$|\mathcal{R}_{j2}(t, s)| + |\mathcal{R}_{j3}(t, s)| \lesssim (2^{jm_2})^{-\frac{n}{k}}.$$

Hence,

$$|\mathcal{R}_j(t, s)| \leq |\mathcal{R}_{j2}(t, s)| + |\mathcal{R}_{j3}(t, s)| + |H_{j3}(t, s)| \lesssim (2^{jm_2})^{-\frac{n}{k}}.$$

Notice that by the assumptions on the supports of Γ and σ_1 , one has $2^j \lesssim t^{\frac{1}{m_2}}$. Then if $n + \beta + \gamma - \frac{m_2 n}{k} \geq 0$,

$$\begin{aligned} & t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\ & \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} |\mathcal{R}_j(t, s)| \\ & \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma - \frac{m_2 n}{k})} \\ & \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} t^{\frac{1}{m_2}(n+\beta+\gamma - \frac{m_2 n}{k})} \lesssim t^{-\frac{n}{k}}, \end{aligned} \quad (3.30)$$

and if $n + \beta + \gamma - \frac{m_2 n}{k} < 0$,

$$\begin{aligned}
& t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} |\mathcal{R}_j(t, s)| \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma - \frac{m_2 n}{k})} \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} 2^{-\frac{n+\beta+\gamma}{m_2} j m_2} \\
& \lesssim t^{-\frac{n+\beta+\gamma}{m_2}} = t^{-\frac{n}{m_2}(\frac{1}{p} + \frac{\gamma}{n})}, \tag{3.31}
\end{aligned}$$

which combined with (3.9), (3.18) and (3.25) lead to

$$\begin{aligned}
\sup_x \|\Omega_{x,2}(t, \cdot)\|_{\dot{\Lambda}^\beta} & \lesssim \sup_x t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_1 \cup E_2 \cup E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \\
& \lesssim t^{\max\{-\frac{n}{k}, -\frac{n}{m_2}(\frac{1}{p} + \frac{\gamma}{n})\}}. \tag{3.32}
\end{aligned}$$

Then we have

$$\begin{aligned}
\|I^\gamma S_{12}(t)f\|_{L^\infty} & \leq \sup_x \|\Omega_{x,2}(t, \cdot)\|_{\dot{\Lambda}^\beta} \|f\|_{H^p} \\
& \lesssim \sup_x t^{-\frac{n+\beta+\gamma}{m_2}} \sum_{j \in E_1 \cup E_2 \cup E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} e^{i\varphi(\xi, t, x)} H(\xi) d\xi \right| \|f\|_{H^p} \\
& \lesssim t^{\max\{-\frac{n}{k}, -\frac{n}{m_2}(\frac{1}{p} + \frac{\gamma}{n})\}} \|f\|_{H^p}, \tag{3.33}
\end{aligned}$$

which finishes the proof of (3.5) for $t > 4^{m_2}$. $\square$

3.2 Decay estimates for high frequency

Lemma 3.3. *Let $\gamma \geq 0$ and $m_1 \geq 2$ be defined in assumption (H1). Assume that $\frac{m_1}{k} - \frac{\gamma}{n} \geq \frac{1}{p} > 1$. Then we have*

$$\|I^\gamma S_2(t)f\|_{L^\infty} \leq |t|^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})} \|f\|_{H^p}, \quad t \neq 0. \tag{3.34}$$

Moreover, if $\frac{m_1}{k} - \frac{\gamma}{n} \geq 1$, we have

$$\|I^\gamma S_2(t)f\|_{L^\infty} \leq |t|^{-\frac{n}{m_1}(1 + \frac{\gamma}{n})} \|f\|_{L^1}, \quad t \neq 0. \tag{3.35}$$

Proof. Since the proof of (3.34) and (3.35) shares exactly the same arguments, we only establish (3.34) for $t > 0$. Similarly to the proof of Lemmas 3.1 and 3.2, we have

$$\|I^\gamma S_2(t)f\|_{L^\infty} \leq \sup_x \|\Omega_x(t, \cdot)\|_{\dot{\Lambda}^\beta} \|f\|_{H^p}, \quad (3.36)$$

where $\beta = n(\frac{1}{p} - 1)$ and

$$\Omega_x(t, y) = \int_{\mathbb{R}^n} e^{itP(\phi(\xi))} \sigma_2(\phi(\xi)) \phi(\xi)^\gamma e^{i\langle x-y, \xi \rangle} d\xi.$$

Let Γ be a standard smooth bump function introduced in Section 2. By (2.3) and changing of variable, we have

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \|\Omega_x(t, \cdot)\|_{\dot{\Lambda}^\beta} &\leq \sup_{x \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}} \left| 2^{j\beta} \int_{\mathbb{R}^n} \Gamma(2^{-j}\phi(\xi)) e^{itP(\phi(\xi))} \sigma_2(\phi(\xi)) \phi(\xi)^\gamma e^{i\langle x, \xi \rangle} d\xi \right| \\ &= \sup_{x \in \mathbb{R}^n} \sum_{j=0}^{\infty} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j\phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right|, \end{aligned} \quad (3.37)$$

where the phase function ψ is defined by

$$\psi(\xi, t, x) = tP(2^j\phi(\xi)) + \langle x, \xi \rangle 2^j.$$

As in Lemma 3.2, for fixed $|x| \neq 0$, we introduce the following sets

$$\begin{aligned} E_1 &= \{0 \leq j : t2^{j(m_1-1)-10} \geq |x|\}, \\ E_2 &= \{0 \leq j : tn\alpha M 2^{j(m_1-1)+10} \leq |x|\}, \end{aligned}$$

and

$$E_3 = \{0 \leq j : t2^{j(m_1-1)-10} < |x| < tn\alpha M 2^{j(m_1-1)+10}\},$$

where M is a large constant. Then the summation in (3.37) can be controlled by

$$\begin{aligned} &\sum_{j=0}^{\infty} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j\phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ &\lesssim \sum_{j \in E_1 \cup E_2 \cup E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j\phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right|. \end{aligned} \quad (3.38)$$

If $j \in E_1$, each integral on RHS of (3.38) can be written as

$$\begin{aligned} & \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \\ &= \int_{\Sigma} \int_0^\infty G(r) \sigma_2(2^j r) e^{iP_2(r)} dr |\nabla \phi(\xi')|^{-1} d\sigma(\xi'), \end{aligned} \quad (3.39)$$

where $G(r)$ is given by (3.12) and

$$P_2(r) = tP(2^j r) + \langle x, \xi' \rangle 2^j r.$$

It follows from the assumption **(H1)** that for $j \in E_1$,

$$\begin{aligned} |\partial_r P_2(r)| &= |t2^j P'(2^j r) + \langle x, \xi' \rangle 2^j| \\ &\gtrsim |t2^j P'(2^j r)| - |x| |\nabla \phi(\xi')|^{-1} 2^j \\ &\geq t2^{jm_1} - |x| |\nabla \phi(\xi')|^{-1} 2^j \gtrsim t2^{jm_1}, \end{aligned}$$

and

$$|\partial_r^\alpha P_2(r)| \lesssim t2^{jm_1}, \quad \alpha \geq 2.$$

On the other hand, for any integer $m \geq 0$, we obtain

$$\begin{aligned} \left| \partial_r^m \frac{1}{P_2'(r)} \right| &= \left| \frac{\sum_\ell C_\ell P_2'(r)^{\alpha_\ell^1} P_2''(r)^{\alpha_\ell^2} \dots P_2^{(m+1)}(r)^{\alpha_\ell^{m+1}}}{P_2'(r)^{m+1}} \right| \\ &\lesssim (t2^{jm_1})^{-1}, \end{aligned} \quad (3.40)$$

where $\alpha_\ell^1 + \alpha_\ell^2 + \dots + \alpha_\ell^{m+1} = m$, $\alpha_\ell^1 + 2\alpha_\ell^2 + \dots + (m+1)\alpha_\ell^{m+1} = 2m$ and C_ℓ is constant for each ℓ . Moreover, for any integer $k \geq 0$

$$|\partial_r^k G(r) \sigma_2(2^j r)| \lesssim 1. \quad (3.41)$$

Let the operator D_r be defined by (3.16). It follows from (3.40) and (3.41) that for any integer $N \geq 0$,

$$\begin{aligned} & |D_r^N (G(r) \sigma_2(2^j r))| \\ &= \left| \left(-\frac{1}{i} \right)^N \sum_{m=0}^N \sum_{l_1, \dots, l_N \in \Lambda_m^N} C_{m,N} \prod_{i=1}^N \partial_r^{l_i} \frac{1}{P_2'(r)} \partial_r^{N-m} G(r) \sigma_2(2^j r) \right| \\ &\lesssim (t2^{jm_1})^{-N}, \end{aligned}$$

where $\Lambda_m^N = \{l_1, \dots, l_N \in \mathbb{Z}^+ : 0 \leq l_1 < \dots < l_N \leq N, l_1 + \dots + l_N = m\}$. Then by integration by parts, we have for any $N \in \mathbb{Z}_+$,

$$\left| \int_0^\infty e^{iP_2(r)} G(r) \sigma_2(2^j r) dr \right| = \left| \int_0^\infty e^{iP_2(r)} D_r^N (G(r) \sigma_2(2^j r)) dr \right| \lesssim (t2^{jm_1})^{-N}, \quad (3.42)$$

which implies that

$$\begin{aligned} & \sum_{j \in E_1} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ & \leq \left(\sum_{j \in E_1 \cap \{t2^{m_1 j} < 1\}} + \sum_{j \in E_1 \cap \{t2^{m_1 j} \geq 1\}} \right) 2^{j(n+\beta+\gamma)} \int_{\mathbb{R}^n} |\Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi))| \phi(\xi)^\gamma d\xi \\ & \lesssim \sum_{j \in E_1 \cap \{t2^{m_1 j} < 1\}} 2^{j(n+\beta+\gamma)} + \sum_{j \in E_1 \cap \{t2^{m_1 j} \geq 1\}} 2^{j(n+\beta+\gamma)} (t2^{jm_1})^{-N} \\ & \lesssim t^{-\frac{n+\beta+\gamma}{m_1}} = t^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})}, \end{aligned} \quad (3.43)$$

where N is chosen to be a large constant. Similar argument as above can be applied for $j \in E_2$ to obtain

$$\begin{aligned} & \sum_{j \in E_2} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ & \lesssim \left(\sum_{j \in E_2 \cap \{t2^{m_1 j} < 1\}} + \sum_{j \in E_2 \cap \{t2^{m_1 j} \geq 1\}} \right) 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ & \lesssim t^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})}. \end{aligned} \quad (3.44)$$

Now turning to $j \in E_3$, noticed that

$$\frac{1}{m_1-1} \left(\log_2 \frac{|x|}{t n \alpha M} - 10 \right) < j < \frac{1}{m_1-1} \left(\log_2 \frac{|x|}{t} + 10 \right),$$

we conclude that E_3 has finite elements, that is $|E_3| \leq K_2$ with K_2 depending on m_1 and M . It is easy to see that when $t2^{m_1 j} \leq 1$,

$$\begin{aligned} & \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ & \lesssim \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} \lesssim t^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})}. \end{aligned} \quad (3.45)$$

On the other hand, when $t2^{m_1j} > 1$, we shall utilize the same procedure as the one in the proof of Lemma 3.2. To do so, write

$$\begin{aligned} & \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \\ &= \int_0^\infty |G(r) \sigma_2(2^j r) e^{itP(2^j r)} \left(\int_\Sigma e^{i2^j r s \langle \omega, \xi' \rangle} \frac{d\sigma(\xi')}{|\nabla \phi(\xi')|} \right) dr \\ &:= I_j(t, s), \end{aligned} \quad (3.46)$$

where $s = |x|$, $x = s\omega$ and $d\sigma(\xi')$ is the Lebesgue induced surface measure on Σ . By using the (3.26), we further have

$$\begin{aligned} I_j(t, s) &= \int_0^\infty G(r) \sigma_2(2^j r) e^{itP(2^j r)} \left(\int_\Sigma e^{i2^j r s \langle \omega, \xi' \rangle} \frac{d\sigma(\xi')}{|\nabla \phi(\xi')|} \right) dr \\ &= \int_0^\infty G(r) \sigma_2(2^j r) e^{itP(2^j r) + i2^j r s \langle \omega, \xi'_+ \rangle} H_+(2^j r s) dr \\ &\quad + \int_0^\infty G(r) \sigma_2(2^j r) e^{itP(2^j r) + i2^j r s \langle \omega, \xi'_- \rangle} H_-(2^j r s) dr \\ &\quad + \int_0^\infty |G(r) \sigma_2(2^j r) e^{itP(2^j r)} H_\infty(2^j r s) dr \\ &:= I_{j1}(t, s) + I_{j2}(t, s) + I_{j3}(t, s). \end{aligned} \quad (3.47)$$

We denote

$$R_{j1}(t, r, s) = tP(2^j r) + 2^j r s \langle \omega, \xi'_+ \rangle$$

in I_{j1} , it follows from the assumption **(H1)** and (3.27) that for $j \in E_3$,

$$|\partial_r^2 R_{j1}(t, r, s)| = |t2^{2j} P''(2^j r)| \gtrsim t2^{jm_1}$$

and for $N \geq 0$,

$$|\partial_r^N (G(r) \sigma_2(2^j r) H_+(2^j r s))| \lesssim (2^j s)^{-\frac{n-1}{k}},$$

which further combined with Lemma 2.2 implies that

$$\begin{aligned} |I_{j1}(t, s)| &\leq (t2^{jm_1})^{-\frac{1}{2}} (\|G(r) \sigma_2(2^j r) H_+(2^j r s)\|_\infty \\ &\quad + \|\partial_r (G(r) \sigma_2(2^j r) H_+(2^j r s))\|_{L^1}) \\ &\lesssim (t2^{jm_1})^{-\frac{1}{k}} (2^j s)^{-\frac{n-1}{k}} \\ &\lesssim (t2^{jm_1})^{-\frac{1}{k}} (2^j t2^{j(m_1-10)})^{-\frac{n-1}{k}} \lesssim (t2^{jm_1})^{-\frac{n}{k}}. \end{aligned} \quad (3.48)$$

Similarly,

$$|I_{j2}(t, s)| + |I_{j3}(t, s)| \lesssim (t2^{jm_1})^{-\frac{n}{k}}.$$

Then we obtain

$$|I_j(t, s)| \leq |I_{j1}(t, s)| + |I_{j2}(t, s)| + |I_{j3}(t, s)| \lesssim (t2^{jm_1})^{-\frac{n}{k}},$$

which further combined with (3.46) and the assumption that $\frac{m_1}{k} - \frac{\gamma}{n} \geq \frac{1}{p} > 1$ implies that

$$\begin{aligned} & \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \\ & \leq \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} |I_j(t, s)| \lesssim \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} (t2^{jm_1})^{-\frac{n}{k}} \\ & \lesssim \sum_{j \in E_3} 2^{j(n+\beta+\gamma)} (t2^{jm_1})^{-\frac{n+\beta+\gamma}{m_1}} \lesssim t^{-\frac{n+\beta+\gamma}{m_1}} = t^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})}. \end{aligned} \quad (3.49)$$

Similarly to the proof of Lemma 3.2, by (3.37), (3.38), (3.46) and (3.49), we obtain

$$\begin{aligned} \|I^\gamma S_2(t)f\|_{L^\infty} & \leq \sup_x \|\Omega_x(t, \cdot)\|_{\dot{\Lambda}^\beta} \|f\|_{H^p} \\ & \lesssim \sup_x \sum_{j \in E_1 \cup E_2 \cup E_3} 2^{j(n+\beta+\gamma)} \left| \int_{\mathbb{R}^n} \Gamma(\phi(\xi)) \sigma_2(2^j \phi(\xi)) \phi(\xi)^\gamma e^{i\psi(\xi, t, x)} d\xi \right| \|f\|_{H^p} \\ & \leq |t|^{-\frac{n}{m_1}(\frac{1}{p} + \frac{\gamma}{n})} \|f\|_{H^p}. \end{aligned}$$

Hence we finish the proof. $\square$

3.3 The proof of the main theorem

In this subsection, we prove Theorem 1.1, i.e., the $H^p - H^q$ estimate for $e^{itP((\phi(\xi)))}$ for $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_{m_1, \gamma}$. The following result concerns the estimate at vertex B_5 in Fig. 6.

Proposition 3.1. *Let m_1 and m_2 be defined by assumptions (H1) and (H2), respectively. Assume that $m_1 \geq 2$, $m > 0$ and $\gamma \geq 0$. Denote by $m = \max\{m_1, m_2\}$, then we have for $\frac{m_1}{k} - \frac{\gamma}{n} \geq \frac{1}{p} > 1$*

$$\|I^\gamma S(t)f\|_{L^\infty} \leq |t|^{-\frac{n}{m}(\frac{1}{p} + \frac{\gamma}{n})} \|f\|_{H^p},$$

and for $\frac{m_1}{k} - \frac{\gamma}{n} \geq 1$,

$$\|I^\gamma S(t)f\|_{L^\infty} \leq |t|^{-\frac{n}{m}(1 + \frac{\gamma}{n})} \|f\|_{L^1}.$$

Proof. Recall that the operator $S(t) = e^{itP(\phi(D))}$ was rewritten as

$$S(t) = S_{11}(t) + S_{12}(t) + S_2(t).$$

By Lemmas 3.1, 3.2 and 3.3, we obtain

$$\begin{aligned} \|S(t)f\|_{L^\infty} &\leq \|S_{11}(t)f\|_{L^\infty} + \|S_{12}(t)f\|_{L^\infty} + \|S_2(t)f\|_{L^\infty} \\ &\leq ((1+|t|)^{-\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n})} + (1+|t|)^{\max\{-\frac{n}{m_2}(\frac{1}{p}+\frac{\gamma}{n}), -\frac{n}{k}\}} + |t|^{-\frac{n}{m_1}(\frac{1}{p}+\frac{\gamma}{n})}) \|f\|_{H^p} \\ &\leq |t|^{-\frac{n}{m}(\frac{1}{p}+\frac{\gamma}{n})} \|f\|_{H^p}. \end{aligned}$$

The L^1-L^∞ estimate can be treated similarly as above, we omit the details here. $\square$

Now we turn to prove Theorem 1.1.

Proof of Theorem 1.1. By assumptions, one has obviously $m > k$. Let

$$\delta = n\left(\frac{m_1}{k} - 1\right) - \gamma$$

and then we have $\frac{m_1}{k} - \frac{\gamma+\delta}{n} \geq 1$. It follows from Proposition 3.1 that

$$\begin{aligned} \|I^\gamma S(t)f\|_{L^\infty} &= \|\phi(D)^\delta I^\gamma S(t)\phi(D)^{-\delta} f\|_{L^\infty} \\ &\lesssim |t|^{-\frac{n}{m}(1+\frac{\delta+\gamma}{n})} \|\phi(D)^{-\delta} f\|_{L^1} \\ &\leq |t|^{-\frac{nm_1}{km}} \|\phi(D)^{-\delta} f\|_{H^1}. \end{aligned}$$

Let $m(\xi) := (\frac{|\xi|}{\phi(\xi)})^\delta$, it follows that m is a homogeneous function degree zero on $\mathbb{R}^n$ that is smooth and positive away from the origin. By Euler's homogeneous formula, $D^\mu m(\xi)$ is a homogeneous function degree $-|\mu|$ that is bounded on sphere $\mathbb{S}^{n-1}$, then we have

$$|\xi|^\mu D^\mu m(\xi) = \left| D^\mu m\left(\frac{\xi}{|\xi|}\right) \right| \lesssim 1,$$

where $D^\mu = \partial_{\xi_1}^{\mu_1} \partial_{\xi_2}^{\mu_2} \cdots \partial_{\xi_n}^{\mu_n}$ and $\mu_1 + \mu_2 + \cdots + \mu_n = |\mu|$. By Lemma 2.3, we know that $m(\xi) \in \mathcal{M}(H^1, H^1)$, hence

$$\begin{aligned} \|I^\gamma S(t)f\|_{L^\infty} &\lesssim |t|^{-\frac{nm_1}{km}} \|\phi(D)^{-\delta} f\|_{H^1} \\ &= |t|^{-\frac{nm_1}{km}} \left\| \left(\frac{|\nabla|}{\phi(D)} \right)^\delta |\nabla|^{-\delta} f \right\|_{H^1} \\ &\lesssim |t|^{-\frac{nm_1}{km}} \| |\nabla|^{-\delta} f \|_{H^1} \\ &= |t|^{-\frac{nm_1}{km}} \|f\|_{H_{-\delta}^1} \\ &\lesssim |t|^{-\frac{nm_1}{km}} \|f\|_{\dot{F}_{1,2}^{-\delta}}. \end{aligned} \tag{3.50}$$

On the other hand, since ϕ is a positive smooth homogeneous function of degree one, $m_1(\xi) := (\frac{\phi(\xi)}{|\xi|})^\gamma$ is a bounded function and hence is a L^2 multiplier, then we have

$$\|I^\gamma S(t)f\|_{L^2} \lesssim \|\phi(D)^\gamma f\|_{L^2} \lesssim \|f\|_{L^\gamma_2} \lesssim \|f\|_{\dot{F}^\gamma_{2,2}}. \quad (3.51)$$

Let $\frac{1}{q} = \frac{1}{2} - \frac{k\gamma}{2n(m_1-k)}$ and $0 < \theta < 1$ thus we obtain

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

Then we obtain $\theta = 1 - \frac{k\gamma}{n(m_1-k)}$,

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = \frac{1}{2} + \frac{k\gamma}{2n(m_1-k)},$$

and

$$(1-\theta)(-\delta) + \theta\gamma = 0.$$

Noticed $\frac{m_1}{k} - \frac{\gamma}{n} \geq 1$, then $\frac{m}{k} - \frac{\gamma}{n} \geq 1$ also. We obtain that $p \geq 1$ and $q > 1$. By lemma 2.4, we have $[\dot{F}^{-\delta}_{1,2}, \dot{F}^\gamma_{2,2}]_\theta = \dot{F}^0_{p,2} = H^p$. By simple calculation, we obtain

$$-\frac{nm_1}{km}(1-\theta) = -\frac{n}{m}\left(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n}\right).$$

Interpolating (3.50) and (3.51), we have

$$\begin{aligned} \|I^\gamma S(t)f\|_{H^q} &\lesssim |t|^{-\frac{nm_1}{km}(1-\theta)} \|f\|_{H^p} \\ &= |t|^{-\frac{n}{m}\left(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n}\right)} \|f\|_{H^p}. \end{aligned}$$

Thus we obtain the estimate (1.9) at the vertex A_5 . Also, Proposition 3.1 implies that (1.9) holds at vertex B_5 in Fig. 6. So an interpolation yields that (1.9) is true on the line segment from A_5 to B_5 . Now notice that the equation for A_5B_5 is given by

$$l_{A_5B_5}: y = \frac{\frac{1}{2} - \frac{k\gamma}{2n(m_1-k)}}{\frac{1}{2} + \frac{k\gamma}{2n(m_1-k)} - \left(\frac{m_1}{k} - \frac{\gamma}{n}\right)} \left(x - \left(\frac{m_1}{k} - \frac{\gamma}{n}\right)\right).$$

The line $l_{A_5B_5}$ and line $x=1$ intersect at point $C_5: (1, \eta')$ with

$$\eta' = \left(\frac{1}{2} - \frac{k\gamma}{2n(m_1-k)}\right) \frac{\frac{m_1}{k} - \frac{\gamma}{n} - 1}{\frac{m_1}{k} - \frac{\gamma}{n} - \frac{1}{2} - \frac{k\gamma}{n(m_1-k)}}$$

and $\eta + \eta' = 1$, where η is given by (1.10). That is, we have

$$\|I^\gamma S(t)f\|_{H^{\frac{1}{\eta'}}} \lesssim |t|^{-\frac{n}{m}(1-\eta'+\frac{\gamma}{n})} \|f\|_{H^1} \quad (3.52)$$

at vertex B_5 in Fig. 6. by duality,

$$\|I^\gamma S(t)f\|_{BMO} \lesssim |t|^{-\frac{n}{m}(\eta+\frac{\gamma}{n})} \|f\|_{H^{\frac{1}{\eta}}}, \quad (3.53)$$

which implies the estimate (1.9) at the vertex D_5 . Then the theorem is completed by interpolations. $\square$

4 Applications

In this section, we shall apply Theorem 1.1 to some specific equations. Let us first introduce the following free schrödinger equation

$$\begin{cases} i\partial_t u = (-\Delta)^\alpha u, & \alpha > 1, \\ u(0) = u_0. \end{cases} \quad (4.1)$$

This reduces to the group $K(t) := e^{-it(-\Delta)^\alpha}$, which corresponds to $\phi(\xi) = |\xi|$, $P(r) = r^{2\alpha}$, $k=2$, that is, the solution has a low frequency enjoy the same scaling. By simple calculation, we see that $P(r)$ satisfies **(H1)** and **(H2)** with $m_1 = m_2 = 2\alpha$. Let $|\nabla|^\gamma = I^\gamma$, then by using Theorem 1.1, we have the following result which has been proved in [15].

Corollary 4.1. *Let $\alpha > 1$, $\gamma \geq 0$ and $m = \max\{m_1, m_2\} = 2\alpha$. Then we have that for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_{\alpha, \gamma}$,*

$$\|I^\gamma K(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q}+\frac{\gamma}{n})} \|f\|_{H^p}. \quad (4.2)$$

In particular, for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_\alpha$,

$$\|K(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{2\alpha}(\frac{1}{p}-\frac{1}{q})} \|f\|_{H^p}.$$

Moreover, we can also consider the following homogeneous dispersive equation,

$$\begin{cases} i\partial_t u + \phi^m(D)u = 0, \\ u(0) = u_0. \end{cases} \quad (4.3)$$

This reduces to the group $B(t) := e^{-it\phi^m(D)}$, which corresponds to $P(r) = r^m$ with $m > 2$. By simple calculation, we see that $P(r)$ satisfies **(H1)** and **(H2)** with $m_1 = m_2 = m$.

Corollary 4.2. Let $m > 2$, $\gamma \geq 0$, $\frac{m}{k} - \frac{\gamma}{n} \geq 1$, we have that for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{I}_{m,\gamma}$,

$$\|I^\gamma B(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n})} \|f\|_{H^p}.$$

In particular, if we take $\phi(\xi) = \sqrt[n]{\xi_1^m + \dots + \xi_n^m}$ ($m = 2, 4, \dots$), then

$$\|e^{-it(\partial_1^m + \dots + \partial_n^m)} f\|_{H^q} \lesssim |t|^{-\frac{n}{m}(\frac{1}{p} - \frac{1}{q})} \|f\|_{H^p}.$$

Notice that $P(r)$ of schrödinger equation (4.1) and (4.3) is homogeneous. Our results can also apply to $P(r)$ that scales differently in high and low frequency. We give here a simply equation

$$\begin{cases} i\partial_t u = (-\Delta)^a u + (-\Delta)^b u, \\ u(0) = u_0. \end{cases} \quad (4.4)$$

This reduce to the group

$$L(t) := e^{-it[(-\Delta)^a + (-\Delta)^b]},$$

which corresponds to $\phi(\xi) = |\xi|$, $P(r) = r^{2a} + r^{2b}$. When $a = 1$ and $b = 2$, we should be familiar with the fourth-order schrödinger equation. In [17], the authors obtained that when the semigroup $U(t) := e^{-it(\Delta^2 - \Delta)}$ satisfies $2 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\delta = \frac{1}{2} - \frac{1}{p}$, $-2n\delta \leq s' - s$ and p' is conjugate of p , the following estimate holds

$$\|U(t)f\|_{B_{p,q}^s} \lesssim k(t) \|g\|_{B_{p',q}^{s'}},$$

where

$$k(t) = \begin{cases} |t|^{\frac{1}{4}\min(s' - s - 2n\delta, 0)}, & |t| \leq 1, \\ |t|^{-n\delta}, & |t| \geq 1. \end{cases}$$

In the following, we use the theorem 1.1 to obtain the $H^p - H^q$ estimate of such semigroups. By simple calculation, we see that $S(r)$ satisfies **(H1)** and **(H2)** with $m_1 = \max\{a, b\}$, $m_2 = \min\{a, b\}$. Let $|\nabla|^\gamma = I^\gamma$, Then by Theorem 1.1, we have

Corollary 4.3. Let $\max\{a, b\} > 2$, $\gamma \geq 0$ and $m = \max\{m_1, m_2\} = \max\{a, b\}$. Then we have that for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{I}_{\max\{a,b\},\gamma}$,

$$\|I^\gamma L(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{\max\{a,b\}}(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n})} \|f\|_{H^p}. \quad (4.5)$$

In particular, for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{I}_{\max\{a,b\}}$,

$$\|L(t)f\|_{H^a} \lesssim |t|^{-\frac{n}{\max\{a,b\}}(\frac{1}{p} - \frac{1}{q})} \|f\|_{H^p}.$$

We can also consider the following equation

$$\begin{cases} i\partial_t u + P_\ell(\phi(D))u = 0, \\ u(0) = u_0, \end{cases} \quad (4.6)$$

where $P_\ell(\xi) = \sum_{j=2}^{\ell} C_j \xi^j$ ($\ell > 2$) is a polynomial and ϕ satisfies **(H_Σ)**. The propagator is given by $E(t) := e^{-itP_\ell(\phi(D))}$. By simple calculation, we see that $P_\ell(\xi)$ satisfies **(H1)** and **(H2)** with $m_1 = \ell, m_2 = 2$. Then by Theorem 1.1, we have

Corollary 4.4. *Let $\gamma \geq 0$, $\frac{\ell}{k} - \frac{\gamma}{n} \geq 1$, $m = \max\{\ell, 2\} = \ell$, Then we have that for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_{\ell, \gamma}$,*

$$\|I^\gamma E(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{\ell}(\frac{1}{p} - \frac{1}{q} + \frac{\gamma}{n})} \|f\|_{H^p}.$$

In particular, for any $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{J}_\ell$,

$$\|E(t)f\|_{H^q} \lesssim |t|^{-\frac{n}{\ell}(\frac{1}{p} - \frac{1}{q})} \|f\|_{H^p}.$$

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