Ann. Appl. Math.
doi: 10.4208/aam.OA-2025-0003

Vol. **41**, No. 1, pp. 112-154
February 2025

# An Extension of the $r^p$ Method for Wave Equations with Scale-Critical Potentials and First-Order Terms

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Received 8 January 2025; Accepted (in revised version) 31 March 2025

Dedicated to the celebration of the 70th birthday of Professor Avy Soffer

**Abstract.** The $r^p$ method, first introduced in [9], has become a robust strategy to prove decay for wave equations in the context of black holes and beyond. In this note, we propose an extension of this method, which is particularly suitable for proving decay for a general class of wave equations featuring a scale-critical time-dependent potential and/or first-order terms of small amplitude. Our approach consists of absorbing error terms in the $r^p$-weighted energy using a novel Grönwall argument, which allows a larger range of $p$ than the standard method. A spherically symmetric version of our strategy first appeared in [22] in the context of a weakly charged scalar field on a black hole whose equations also involve a scale-critical potential.

**AMS subject classifications**: 35

**Key words**: Scale-critical potential, $r^p$ method.

## 1 Introduction

The $r^p$ method of Dafermos–Rodnianski [9] is a versatile tool to prove decay in time for solutions $\phi$ to waves equations of the form $\Box_g\phi=0$, which is sufficiently robust for applications to nonlinear wave equations (see, e.g., [7, 8, 19, 24]). The

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method is carried out in physical space and relies on the radiative structure of wave equations on an asymptotically flat spacetime $g$, for which $r\phi$ (usually) admits a finite limit–the Friedlander radiation field $\psi_{\mathcal{I}}(u,\omega)$–defined, for $u\in\mathbb{R}$, $\omega\in\mathbb{S}^2$ as

$$\psi_{\mathcal{I}}(u,\omega):=\lim_{v\to+\infty} r\phi(u,v,\omega), \tag{1.1}$$

where $u$ and $v$ are respectively retarded-time and advanced-time coordinates, corresponding to $u=t-r$ and $v=t+r$ on the usual Minkowski spacetime. In its simplest expression, the key idea of the $r^p$ method is to exploit the boundedness of a $r^p$ weighted energy of the following form, for $0\le p\le 2$:

$$\sup_u \sum_{|\beta|\le 2}\int_{v=v_0}^{+\infty}\int_{\mathbb{S}^2} r^p|\nabla_{\mathbb{S}^2}^{\beta}\partial_v(r\phi)|^2(u,v,\omega)dvd\omega, \tag{1.2}$$

to obtain pointwise decay in time of $\phi$ at the rate $u^{-\frac{p}{2}}$ as $u\to+\infty$, under the additional conditions that

I. Energy boundedness (in the style of (1.8) below) holds.

II. An integrated local decay estimate (in the style of (1.9)), also known as a Morawetz estimate, is valid.

In this paper, our goal is to apply the $r^p$ method and obtain decay in time estimates for the following class of linear wave equations with scale-critical potential and/or scale-critical first-order terms of small amplitude.

$$\Box_g\phi=\frac{1}{r^2(u,v)}\left(\sum_{i=0}^{1}[\epsilon w_i(u,v)+W_i(u,v)]\cdot\partial_u^i\phi+[\epsilon q(u,v)+Q(u,v)]\cdot r\partial_v\phi\right), \tag{1.3a}$$

$$g=-\Omega^2(u,v)dudv+r^2(u,v)d\sigma_{\mathbb{S}^2}, \tag{1.3b}$$

where $\epsilon\in\mathbb{R}$ is a small constant, and $g$ is a spherically-symmetric and asymptotically flat[†] Lorentzian metric in the mild sense that

$$\begin{aligned}&|1+\partial_u r(u,v)|,\ |1+\partial_v r(u,v)|,\ |\Omega^2(u,v)-4|,\ r|\partial_v\Omega^2|(u,v)\\ &\lesssim r^{-1}(u,v) \qquad\qquad \text{as}\quad v\to+\infty,\end{aligned} \tag{1.4a}$$

$$\begin{aligned}&|\partial_v^2 r|(u,v),\ |\partial_u\partial_v r|(u,v),\ |\partial_u\partial_v^2 r|(u,v)\\ &\lesssim r^{-2}(u,v) \qquad\qquad \text{as}\quad v\to+\infty.\end{aligned} \tag{1.4b}$$

[†]As we will discuss in Section 1.3, the usual Minkowski metric $m=-dt^2+dx^2+dy^2+dz^2$, associated to $\Box_m=-\partial_t^2+\partial_x^2+\partial_y^2+\partial_z^2$, satisfies (1.4), together with many other usual spacetime metrics $g$, such as the Schwarzschild spacetime.