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Numerical Approaches to Compute Spectra of Non-Self Adjoint Operators in Two and Three Dimensions

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Abstract. In this article we are interested in the numerical computation of spectra of non-self adjoint quadratic operators, in two and three spatial dimensions. Indeed, in the multidimensional case very few results are known on the location of the eigenvalues. This leads to solve nonlinear eigenvalue problems. In introduction we begin with a review of theoretical results and numerical results obtained for the one dimensional case. Then we present the numerical methods developed to compute the spectra (finite difference discretization) for the two and three dimensional cases. The numerical results obtained are presented and analyzed. One difficulty here is that we have to compute eigenvalues of strongly non-self-adjoint operators which are unstable. This work is in continuity of a previous work in one spatial dimension [3].

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1 Introduction

The aim of this paper is the extension of [3] (one dimensional in space) to higher dimensional spaces.

In this work we study the polynomial family of operators

$$L(\lambda)=H_0+\lambda H_1+\cdots+\lambda^{m-1}H_{m-1}+\lambda^m\mathbb{I},$$

where $H_0,H_1,\cdots,H_{m-1}$ are operators defined on some Hilbert space $\mathcal{H}$, $\mathbb{I}$ is the identity operator and λ is a complex parameter. We are interested to study the spectrum of the family $L(\lambda)$.

The problem $L(\lambda)u(x)=0$, is called a non-linear eigenvalue problem for $m\geq 2$. The number $\lambda_0\in\mathcal{C}$ is called an eigenvalue of $L(\lambda)$, if there exists $u_0\in\mathcal{H}$, $u_0\neq 0$ such that $L(\lambda_0)u_0=0$. We consider here a quadratic family ($m=2$) and in particular we are interested in the case

$$L_P(\lambda)=\Delta+(P(x)\lambda)^2,$$

which is defined on the Hilbert space $\mathcal{L}^2(\mathbb{R}^n)$, where P is an elliptic positive polynomial of degree $M\geq 2$. For this example the known results for the existence of eigenvalues are given for $n=1$ and n is even (see [20]).

The main goal of the work of [1] and [2] is to check the following conjecture, stated by Helffer-Robert-Wang [20] : *for every dimension n, for every $M\geq 2$, the spectrum of L_P is non empty*, which is confirmed for the following cases :

- $n=1,3$, for every polynomial P of degree $M\geq 2$;
- $n=5$, for every convex polynomial P satisfying some technical conditions;
- $n=7$, for every convex polynomial P.

This result extends to the case of quasi-homogeneous polynomial and quasi-elliptic, for example $P(x,y)=x^2+y^4$, $x\in\mathbb{R}^{n_1}$, $y\in\mathbb{R}^{n_2}$, $n_1+n_2=n$, and n is even. These last results were proved by computing the coefficients of a semi-classical trace formula and by using the theorem of Lidskii.

In [3], some theoretical results are presented for $n=1$, for quadratic family of operators :

$$L(\lambda)=L_0+\lambda L_1+\lambda^2\mathbb{I},$$

where L_0 and L_1 are operators on an Hilbert space and $\mathbb{I}$ is the identity operator. Moreover some numerical methods are presented to compute the spectrum of such operators with a non self-adjoint linear eigenvalue problem, in one dimension in space. The proposed numerical methods are based on spectral methods and finite