Ann. Appl. Math.
doi: 10.4208/aam.OA-2024-0019

Vol. **41**, No. 2, pp. 176-193
May 2025

A Hybrid Conjugate Gradient Method with Trust Region for Large-Scale Unconstrained Optimization Problems

A. P. Byengonzi[1], P. Kaelo[1,*], M. Koorapetse[1] and P. Mtagulwa[2]

[1] *Department of Mathematics, University of Botswana, P/Bag UB00704, Gaborone, Botswana*

[2] *Department of Mathematics, Botho University, P.O. Box 501564, Gaborone, Botswana*

Received 25 September 2024; Accepted (in revised version) 23 February 2025

Abstract. In this work, we modify a conjugate gradient (CG) method recently proposed in the literature, where a PRP conjugate gradient method is modified using trust region. Particularly, we propose a hybrid CG method that incorporates the parameters β^{PRP}, β^{FR} and β^{CD}, and this new search direction satisfies both the trust region feature and the sufficient descent conditions. Furthermore, under suitable conditions the developed method is proved to be globally convergent. The method is tested on some benchmark problems from the literature and numerical results show that it is quite efficient in solving large scale problems.

AMS subject classifications: 90C06, 90C30, 65K05

Key words: Conjugate gradient method, global convergence, strong Wolfe line search, trust-region.

---

1 Introduction

We consider unconstrained optimization problems of the form

$$\min\ f(x), \tag{1.1}$$

*Corresponding author.
Email: kaelop@ub.ac.bw (P. Kaelo)

where $f:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable. Newton, conjugate gradient, Quasi-Newton and steepest descent [16,18,21] are some of the methods that have been used to solve (1.1), with conjugate gradient (CG) methods being the most popular. They have been popular because of their low-cost memory requirements, computational efficiency and strong convergence properties. Conjugate gradient methods have also been applied to many fields, such as reconstruction of radial magnetic resonance (MR) images [38], portfolio selection [1, 8, 12, 33], motion control problems [2, 3], compressive sensing [35] and image restoration problems [37].

Given an initial guess $x_0\in\mathbb{R}^n$, conjugate gradient methods generate iterations

$$x_{k+1}=x_k+\alpha_k d_k,\quad k=0,1,2,\cdots, \tag{1.2}$$

where x_k is the current iteration, $\alpha_k>0$ is the step length, usually computed using inexact line search conditions, and d_k is the search direction defined by

$$d_k=\begin{cases}-g_k, & k=0,\\ -g_k+\beta_k d_{k-1}, & k\geq 1,\end{cases} \tag{1.3}$$

where $g_k=\nabla f(x_k)$ and β_k is a CG parameter. Conjugate gradient methods differ according to the choice of β_k. Each method tends to have unique numerical and convergence performance under inexact line search conditions. Some common β_k formulas are known for their strong convergence properties but tend to have poor numerical performance due to often jamming. These include the Fletcher and Reeves (FR) [15], Dai and Yuan (DY) [11] and Conjugate Descent (CD) [14]

$$\beta_k^{FR}=\frac{\|g_k\|^2}{\|g_{k-1}\|^2},\quad \beta_k^{DY}=\frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}},\quad \beta_k^{CD}=-\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}},$$

where $\|\cdot\|$ denotes the Euclidean norm, and $y_{k-1}=g_k-g_{k-1}$. On the other hand, Hestenes and Stiefel (HS) [19], Polak-Ribière-Polyak (PRP) [28, 29] and Liu and Storey (LS) [27]

$$\beta_k^{HS}=\frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}},\quad \beta_k^{PRP}=\frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2},\quad \beta_k^{LS}=-\frac{g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}},$$

have shown to have better numerical performance but may fail to converge.

In order to improve their performance, variations of CG methods such as three-term conjugate gradient, spectral conjugate gradient and hybrid conjugate gradient methods have been proposed [1, 2, 8, 22, 23, 34, 37, 38]. Hybrid conjugate gradient methods are formulated by combining different CG methods, hence taking advantage of the unique properties that each β_k excels in, and thus avoiding jamming and/or