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Stability and Convergence Analysis of a Linear Energy Stable Scheme for a Cahn-Hilliard Model with Smooth or Weakly Singular Non-Local Term

Moumita Mandal[1], Manisha Chowdhury[1] and Jie Shen[2,*]

[1] *Department of Mathematics, Indian Institute of Technology Jodhpur, Rajasthan 342 030, India*
[2] *School of Mathematical Science, Eastern Institute of Technology, Ningbo, Zhejiang 315200, China*

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Abstract. We consider a Cahn-Hilliard gradient flow model with a free energy functional, which contains a non-local term in addition to linear and non-linear local terms. The non-local terms can be based on smooth and weakly singular kernel operators. We establish the well-posedness of this problem, construct an unconditional energy stable scheme, and carry out a stability and convergence analysis. Several numerical results are presented to illustrate the efficiency and robustness of the proposed scheme.

AMS subject classifications: 65M12, 65M70, 35K35, 35K61

Key words: Cahn-Hilliard, weakly singular, non-local, energy stable, existence and uniqueness, stability and convergence.

1 Introduction

Various physical dissipative systems can be described by gradient flow models. A large number of studies are conducted for gradient flows driven by local free energy (see [16, 17, 27–29, 39, 43] and the references therein), but these local models

*Corresponding author.
Emails: moumita@iitj.ac.in (M. Mandal), chowdhurymanisha8@gmail.com (M. Chowdhury), jshen@eitech.edu.cn (J. Shen)

are inadequate to describe certain dissipative physical phenomena with long-range interactions accurately. There has been a growing interest in modeling such systems using non-local models in recent years. Our interest of study is a class of non-local gradient flow models that are governed by the energy functional

$$E[v]=\int_\Omega F(v)d\boldsymbol{x}+\int_\Omega v(\boldsymbol{x})H(\boldsymbol{x})d\boldsymbol{x}+\frac{1}{2}\int_\Omega\int_\Omega W(\boldsymbol{x},\boldsymbol{y})v(\boldsymbol{x})v(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y},$$

where F is an internal free energy density, H and W denote the confinement and interaction potentials respectively. Such non-local models appear in several branches of sciences, such as material science [21], biological systems (see [5, 36]), physical science (see [10, 11]), etc. The interaction potential W may be smooth or singular in nature. The non-local models with a smooth kernel W are used in studying granular flows (see [8, 11]), interaction of gases (see [10, 38]), biological aggregation (see [24, 36]), peridynamics [30], quasi-crystallization (see [2, 6, 20]) etc. Meanwhile, the singular potentials are mostly used in modeling the interaction energy associated with a repulsive-attractive potential in the study of various atomic structures. Such potentials are known as repulsive-attractive Morse or power-law potentials [14]. The studies in (see [3, 12, 36]) cover a variety of non-local interaction models with such power-law potentials. The singular potentials make the interaction energy infinite, and various interesting questions in the study of crystallization are raised with these potentials, e.g., in [34], an asymptotic behavior of the ground state energy of many-particle systems is mathematically analyzed, where the interaction energy is represented by a singular potential kernel.

In this paper, we consider a Cahn-Hilliard gradient flow model governed by the non-local free energy containing both smooth and weakly singular potentials. It is generally desirable that numerical methods for such systems preserve the discrete energy dissipation law. In recent years, several popular numerical approaches are proposed to construct energy stable schemes for gradient flows with nonlinear (local) density function, for example, convex splitting method [17], stabilization method (see [26, 43]) and invariant energy quadratization (IEQ) method (see [39, 42]), and with non-local free energies containing only smooth interaction potentials, for example, the Discontinuous Galerkin (DG) method [31], Primal-Dual methods [14], the Back-and-Forth method [19] etc. In particular, the SAV approach [27] enjoys many advantages compared with other approaches. In the convex spitting method [17], one needs to solve nonlinear equations at each time step, which can be expensive, and in the IEQ method, linear systems with variable coefficients need to be solved. At the same time, the SAV scheme requires only the solution of linear systems with constant coefficients at each time step. Besides, the first- and second-order SAV schemes are unconditionally energy-stable and applicable to a large class of gradient