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Entire Sign-Changing Solutions to the Fractional Critical Schrödinger Equation

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Abstract. In this paper, we consider the fractional critical Schrödinger equation (FCSE)

$$(-\Delta)^s u - |u|^{2_s^*-2}u = 0,$$

where $u\in\dot{H}^s(\mathbb{R}^N)$, $N\geq 4$, $0<s<1$ and $2_s^*=\frac{2N}{N-2s}$ is the critical Sobolev exponent of order s. By virtue of the variational method and the concentration compactness principle with the equivariant group action, we obtain some new type of non-radial, sign-changing solutions of (FCSE) in the energy space $\dot{H}^s(\mathbb{R}^N)$. The key component is that we take the equivariant group action to construct several subspace of $\dot{H}^s(\mathbb{R}^N)$ with trivial intersection, then combine the concentration compactness argument in the Sobolev space with fractional order to show the compactness property of Palais-Smale sequences in each subspace and obtain the multiple solutions of (FCSE) in $\dot{H}^s(\mathbb{R}^N)$.

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1 Introduction

This paper is concerned with the existence of the sign-changing solutions for the following fractional critical Schrödinger equation in higher dimensions

$$\begin{cases}(-\Delta)^s u-|u|^{2_s^*-2}u=0 \quad \text{in } \ \mathbb{R}^N,\\ u\in \dot{H}^s\left(\mathbb{R}^N\right),\end{cases} \tag{1.1}$$

where $N\geq 4$, $0<s<1$, $2_s^*=\frac{2N}{N-2s}$ is the critical exponent of order s, $(-\Delta)^s$ denotes the fractional Laplace operator and $\dot{H}^s\left(\mathbb{R}^N\right)$ denotes the real-valued, homogenous Sobolev space, i.e.,

$$\dot{H}^s\left(\mathbb{R}^N\right)=\left\{u\in\mathcal{S}'\left(\mathbb{R}^N\right)\,\Big|\,\|u\|_{\dot{H}^s}<+\infty\right\}, \tag{1.2}$$

with

$$\|u\|_{\dot{H}^s}^2=\int_{\mathbb{R}^N}|\xi|^{2s}|(\mathcal{F}u)(\xi)|^2\mathrm{d}\xi,$$

where $\mathcal{F}u$ denotes the Fourier transform of u:

$$\mathcal{F}u(\xi)=\frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}u(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x.$$

Fractional Schrödinger equations (1.1) arise as models in fractional quantum mechanics, including path integral over the Lévy flights paths, see for instance [27–29], and as Euler-Lagrange equations for the Hardy-Littlewood-Sobolev inequalities, please see [15, 22, 30] and references therein.

The related problem about the positive solutions of (1.1) has attracted lots of attention. Firstly, the existence of positive solutions of (1.1) is related to the existence of extremizers to the Hardy-Littlewood-Sobolev inequalities. Lieb considered the following Hardy-Littlewood-Sobolev inequality in [30]

$$\|u\|^2_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}\leq S(N,s)\|u\|^2_{\dot{H}^s(\mathbb{R}^N)}, \tag{1.3}$$