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Energy Equality for the Isentropic Compressible Navier-Stokes Equations Without Upper Bound of the Density

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Dedicated to the celebration of the 70th birthday of Professor Avy Soffer

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Abstract. In this paper, we are concerned with the minimal regularity of both the density and the velocity for the weak solutions keeping energy equality in the isentropic compressible Navier-Stokes equations. The energy equality criteria without upper bound of the density are established for the first time. Our results imply that the lower integrability of the density ρ means that more integrability of the velocity v or the gradient of the velocity ∇v are necessary for energy conservation of the isentropic compressible fluid and the inverse is also true.

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1 Introduction

The classical isentropic compressible Navier-Stokes equations can be described as

$$\begin{cases} \rho_t+\nabla\cdot(\rho v)=0, \\ (\rho v)_t+\mathrm{div}(\rho v\otimes v)+\nabla P(\rho)-\mathrm{div}\,(\mu\mathfrak{D}v)-\nabla(\lambda\mathrm{div}\,v)=0, \end{cases} \tag{1.1}$$

where ρ stands for the density of the flow, v represents the flow velocity field and $P(\rho)=\rho^\gamma$ is the scalar pressure with the adiabatic index $\gamma>1$; The viscosity coefficients μ and λ satisfy $\mu\geq 0$ and $2\mu+d\lambda>0$; $\mathfrak{D}v=\frac{1}{2}(\nabla v+\nabla v^T)$ is the strain tensor; We complement equations (1.1) with initial data

$$\rho(0,x)=\rho_0(x), \quad \rho v(0,x)=\rho_0 v_0(x), \quad x\in\Omega, \tag{1.2}$$

where we define $v_0=0$ on the set $\{x\in\Omega\colon\,\rho_0=0\}$. In the present paper, we restrict our interest to the bounded domain with periodic boundary condition, that is, $\Omega=\mathbb{T}^d$ with dimension $d\geq 2$.

One of the celebrated results of the isentropic compressible Navier-Stokes equations (1.1) is the global existence of the finite energy weak solutions due to Lions [18] with $\gamma\geq\frac{3d}{d+2}$ for $d=2$ or 3. Subsequently, in [10], Feireisl-Novotny-Petzeltová further extended the Lions' work to $\gamma>\frac{3}{2}$ for $d=3$. In [13], Jiang and Zhang considered the global existence of weak solutions for the case $\gamma>1$ with the spherical symmetric initial data. For the convenience of the reader, we recall the definition of the finite energy weak solution as follows:

Definition 1.1. *A pair (ρ,v) is called a weak solution to* (1.1) *with initial data* (1.2) *if (ρ,v) satisfies*

(i). Eq. (1.1) *holds in $\mathcal{D}'(0,T;\mathbb{T}^d)$ and*

$$P(\rho),\rho|v|^2\in L^\infty(0,T;L^1(\mathbb{T}^d)), \quad \nabla v\in L^2(0,T;L^2(\mathbb{T}^d)), \tag{1.3}$$

(ii). the density ρ is a renormalized solution of (1.1) *in the sense of [9].*

(iii). the energy inequality holds

$$\mathcal{E}(t)+\int_0^T\int_{\mathbb{T}^d}\left(\mu|\nabla v|^2+(\mu+\lambda)|\mathrm{div} v|^2\right)dxdt\leq\mathcal{E}(0), \tag{1.4}$$

where

$$\mathcal{E}(t)=\int_{\mathbb{T}^d}\left(\frac{1}{2}\rho|v|^2+\frac{\rho^\gamma}{\gamma-1}\right)dx.$$