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Global Classical Solution and Asymptotic Behavior to a Kind of Linearly Degenerate Quasilinear Hyperbolic System

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Abstract. This paper is concerned with the global classical solution and the asymptotic behavior to a kind of linearly degenerate quasilinear hyperbolic system in several space variables. When the semilinear terms contain at least two waves with different propagation speeds, we can prove that the system considered admits a global classical solution by the weighted energy estimate under the small and suitable decay assumptions on the initial data. Furthermore, we can show that the solution converges to a solution of the linearized system based on the decay property of the nonlinearities.

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Key words: Hyperbolic system, linearly degenerate, positive weighted energy estimates, global existence, asymptotic behavior.

1 Introduction

Quasilinear hyperbolic systems in several space variables can be described as follows

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{n}A_j(u)\frac{\partial u}{\partial x_j}=B(u), \tag{1.1}$$

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where $u=(u_1,\cdots,u_N)^T$ is the unknown vector function of $(t,x_1,\cdots,x_n)$ and $A_j(u)$ $(j=1,\cdots,n)$ is a $N\times N$ matrix with smooth elements $a_{jkl}(u)$ $(k,l=1,\cdots,N)$, $B(u)$ denotes the source terms. The concepts of linearly degenerate and genuinely nonlinear in high dimensions have been made in the following way (see [18, 19]).

Definition 1.1. *The i-th characteristic field of system* (1.1) *is linearly degenerate, if*

$$\nabla\lambda_i(u,\xi)\cdot r_i(u,\xi)\equiv 0, \quad \forall u\in\Omega, \quad \forall \xi\in\mathbb{S}^{n-1}, \tag{1.2}$$

while it is genuinely nonlinear, if

$$\nabla\lambda_i(u,\xi)\cdot r_i(u,\xi)\neq 0, \quad \forall u\in\Omega, \quad \forall \xi\in\mathbb{S}^{n-1}, \tag{1.3}$$

where $\xi=(\xi_1,\cdots,\xi_n)^T\in\mathbb{S}^{n-1}$, $\lambda_1(u,\xi),\cdots,\lambda_N(u,\xi)$ *are* N *real eigenvalues of* $A(u,\xi)=\sum_{j=1}^n A_j(u)\xi_j$ *and* $\{r_i(u,\xi)\}_{i=1}^N$ *is a complete set of right eigenvectors of* $A(u,\xi)$.

If all the characteristic fields of $A(u,\xi)$ *are linearly degenerate, we call the hyperbolic system* (1.1) *totally linearly degenerate.*

As is known to all, the structures of linearly degenerate and genuinely nonlinear play important roles in the global existence and blowup of solutions for the Cauchy problem of quasilinear hyperbolic systems in dimension one. By the method of characteristics, the global existence of smooth solutions for hyperbolic systems has been proved by Dai and Kong [4] when the system is linearly degenerate and the initial data has compact support and a sufficiently small amplitude. Furthermore, Li, Zhou and Kong [16] and Zhou [31] proved the global existence of smooth solutions to the Cauchy problems of quasilinear hyperbolic systems under the assumption of "weakly linearly degenerate", which is a new concept which contains the case of "linearly degenerate". When the system or part of its characteristic fields is genuinely nonlinear, many authors proved the formation of shocks in finite time, one can refer to [12, 19] and references therein. One can also refer to the book of Li and Wang [15], Kong [11] for a complete survey on the characteristic method.

For the hyperbolic conservation laws in several space dimensions $(n>1)$ with "totally linearly degenerate" characteristics, Majda [19] has conjectured that *the system has global smooth solutions when the initial data are in* $H^s(\mathbb{R}^n)$ $(s>1+\frac{n}{2})$ *unless the solution itself blows up in finite time. In particular the shock wave formation never happens for any smooth initial data.* One typical example with linearly degenerate characteristics is the compressible Euler equation of Chaplygin gas, and Majda's conjecture is proved to be true under some symmetric assumptions. Godin [7] and Ding, Witt and Yin [5] obtained the global radial solutions for the non-isentropic Chaplygin gas when $n=3,2$, respectively. Hou and Yin [9, 10], Wei, Zhang and Zhao [28] obtained the global axisymmetric solution when $n=2$. Adopting different