

Reconfiguration Graphs for Vertex Colorings of P_5 -Free Graphs

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Abstract. For any positive integer k , the reconfiguration graph for all k -colorings of a graph G , denoted by $\mathcal{R}_k(G)$, is the graph where vertices represent the k -colorings of G , and two k -colorings are joined by an edge if they differ in color on exactly one vertex. Bonamy et al. established that for any 2-chromatic P_5 -free graph G , $\mathcal{R}_k(G)$ is connected for each $k \geq 3$. On the other hand, Feghali and Merkel proved the existence of a $7p$ -chromatic P_5 -free graph G for every positive integer p , such that $\mathcal{R}_{8p}(G)$ is disconnected.

In this paper, we offer a detailed classification of the connectivity of $\mathcal{R}_k(G)$ concerning t -chromatic P_5 -free graphs G for cases $t=3$, and $t \geq 4$ with $t+1 \leq k \leq \binom{t}{2}$. We demonstrate that $\mathcal{R}_k(G)$ remains connected for each 3-chromatic P_5 -free graph G and each $k \geq 4$. Furthermore, for each $t \geq 4$ and $t+1 \leq k \leq \binom{t}{2}$, we provide a construction of a t -chromatic P_5 -free graph G with $\mathcal{R}_k(G)$ being disconnected. This resolves a question posed by Feghali and Merkel.

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1 Introduction

Reconfiguration problems, spanning various fields, involve transforming solutions to a source problem into one another through elementary steps. These problems have been studied across various topics in graph theory, including vertex colorings, perfect matchings, independent sets, dominating sets, and more. For further details, readers are referred to surveys by Nishimura [29] and van den Heuvel [26].

In this paper, we study reconfigurations for vertex colorings of graphs. All graphs under consideration are finite and simple. For undefined notation and terminology, we refer readers to [9]. Let $G = (V(G), E(G))$ be a graph, and k be a positive integer. A *proper k -coloring* of G is a mapping $\phi: V(G) \rightarrow \{1, 2, \dots, k\}$ such that $\phi(u) \neq \phi(v)$ for any two adjacent vertices $u, v \in V(G)$. We simply write k -coloring for proper k -coloring in this paper, since all colorings under consideration are proper. Additionally, G is called *k -colorable* if it admits a proper k -coloring. The *chromatic number* of G , denoted by $\chi(G)$, is the smallest integer k such that G is k -colorable. In particular, G is called *k -chromatic* if $\chi(G) = k$. The *reconfiguration graph* for all k -colorings of G , also called the *k -recoloring graph*, denoted by $\mathcal{R}_k(G)$, is the graph whose vertices are the k -colorings of G and two colorings are joined by an edge if they differ in color on exactly one vertex.

As a major problem in this field, the connectivity of $\mathcal{R}_k(G)$ has attracted widespread interest. A result of Jerrum [27] implies the existence of k for each graph G , such that $\mathcal{R}_k(G)$ is connected. Precisely, he proved that $\mathcal{R}_k(G)$ is connected for each integer $k \geq \Delta(G) + 2$. So it is of particular interest to investigate, for a given class $\mathcal{G}$ of graphs, which values of k make $\mathcal{R}_k(G)$ connected for each graph G in $\mathcal{G}$. Notably, it is observed that there exists no direct correlation between the connectivities of graphs $\mathcal{R}_i(G)$ and $\mathcal{R}_j(G)$ for a given graph G , where i and j are two integers with $i > j \geq \chi(G)$ (see Proposition 2.1). Many related results have been proved in some special graphs classes, including degenerate graphs [2, 10–13, 24], planar graphs [1, 15, 17–21, 23], and perfect graphs [8, 14, 24, 25, 28]. In this paper, we focus on P_ℓ -free graphs, which contain no induced path of length $\ell - 1$.

Bonamy and Bousquet [7] proved that, for each $t \geq 1$ and $k \geq t + 1$, $\mathcal{R}_k(G)$ is connected for each t -chromatic P_4 -free graph G . It is worth noting that the lower bound of k is optimal, because any t -coloring of K_t is an isolated vertex in $\mathcal{R}_t(K_t)$, where K_t is the complete graph of order t . A natural question arises: Does the analogue hold for P_ℓ -free graphs for $\ell \geq 5$? This property is trivially satisfied by 1-chromatic P_ℓ -free graphs. But, unfortunately, based on a result of Cereceda, van den Heuvel, and Johnson [14], Bonamy and Bousquet [7] observed that it is not the case for 2-chromatic P_6 -free graphs.