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Implicit Runge-Kutta-Nyström Methods with Lagrange Interpolation for Nonlinear Second-Order IVPs with Time-Variable Delay

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Abstract. This paper deals with nonlinear second-order initial value problems with time-variable delay. For solving this kind of problems, a class of implicit Runge-Kutta-Nyström (IRKN) methods with Lagrange interpolation are suggested. Under the suitable condition, it is proved that an IRKN method is globally stable and has the computational accuracy $\mathcal{O}(h^{\min\{p,\mu+\nu+1\}})$, where p is the consistency order of the method and $\mu,\nu \geq 0$ are the interpolation parameters. Combining a fourth-order compact difference scheme with IRKN methods, an initial-boundary value problem of nonlinear delay wave equations is solved. The presented experiments further confirm the computational effectiveness of the methods and the theoretical results derived in previous.

AMS subject classifications: 65L03, 65L04, 65L80

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1 Introduction

The initial value problems (IVPs) of second-order delay differential equations (DDEs) are a kind of important models for describing the practical scientific phenomena arising in vibration mechanics, mechanical engineering, biodynamics, automatic control and the other related fields (see e.g., [1,2]). Nevertheless, for the IVPs of nonlinear second-order DDEs, it is very difficult to obtain their exact solutions. Hence, in the recent years, ones have begun to develop various numerical methods to solve this kind of problems. For

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example, Papageorgiou & Famelis [3] constructed explicit Runge-Kutta-Nyström methods, Martín & García [4] presented variable-stepsize multistep methods, Ramadan, El-Sherbeiny & Sherif [5] proposed polynomial spline methods, Seong & Majid [6] gave an Adams-Moulton-type method, Kherd, Omar, Saaban & Adeyeye [7] suggested operational matrix methods, Zhang& Wang [8] derived generalized Strömer-Cowell methods and Li & Zhou [9] further adapted the methods in [8] into block generalized Strömer-Cowell methods.

In the existing numerical methods for solving IVPs of second-order DDEs, one-step methods usually have higher computational efficiency than multistep methods with the same consistency order since the computational procedure of an one-step method is self-starting. In view of this, in the actual computation, ones often prefer to use one-step methods to solve the problems. As an example, the mentioned-above explicit Runge-Kutta-Nyström methods are just a type of one-step methods. It is well-known that explicit methods do not work for stiff problems as the boundedness of their stability regions confines the computational stepsize into excessively small and thus leads to an unsuccessful computation. In order to overcome this defect, in the present paper, we will consider implicit Runge-Kutta-Nyström methods with Lagrange interpolation to solve nonlinear second-order IVPs with time-variable delay.

The rest of this paper is organized as follows. In Section 2, by adapting the standard IRKN (see e.g., [10–12]) methods and combining the Lagrange interpolation, we construct a class of new IRKN methods to solve nonlinear second-order IVPs with time-variable delay. In Section 3, we perform an error analysis for IRKN methods and proved that the methods can arrive at the computational accuracy $\mathcal{O}(h^{\min\{p,\mu+\nu+1\}})$ under the suitable condition, where p is the consistency order of the method and $\mu,\nu\geq 0$ are the interpolation parameters. In Section 4, we study nonlinear global stability of IRKN methods and derive a global stability criterion of the methods. In Section 5, with a combination of the fourth-order compact difference scheme and IRKN methods, we present an application to an initial-boundary value problem (IBVP) of nonlinear delay wave equations. The presented numerical experiments further verify the computational effectiveness of the methods and the theoretical results obtained in previous sections.

2 Nonlinear second-order IVPs with time-variable delay and their IRKN methods

Consider the following nonlinear d-dimensional second-order IVPs with time-variable delay $\tau(t) > 0$:

$$y''(t) = f(t, y(t), y(t-\tau(t))), \qquad t \in [t_0, T]; \tag{2.1a}$$

$$y(t) = \varphi(t), \quad y'(t) = \varphi'(t), \qquad t \in [\tau_0, t_0], \tag{2.1b}$$

where functions τ, f and φ are assumed to be sufficiently smooth on their respective domains,

$$\tau_0=\min_{t_0\le t\le T}[t-\tau(t)]$$

and f satisfies condition:

$$\varphi''(t_0)=f(t_0,\varphi(t_0),\varphi(t_0-\tau(t_0)))$$

and the following Lipschitz condition with constants $L_1,L_2>0$:

$$\|f(t,y,z)-f(t,\tilde{y},\tilde{z})\|_\infty\le L_1\|y-\tilde{y}\|_\infty+L_2\|z-\tilde{z}\|_\infty,\quad \forall t\in[t_0,T],\quad y,\tilde{y},z,\tilde{z}\in\mathbb{R}^d. \tag{2.2}$$

Throughout this paper, we always assume that problem (2.1) has a unique solution $y(t)$ smooth enough on $[t_0,T]$ and its derivatives can be bounded by

$$\left\|\frac{d^k y(t)}{dt^k}\right\|_\infty\le M_k,\quad \forall t\in[t_0,T],\quad k\in\mathbb{N}. \tag{2.3}$$

For solving problems (2.1), we take stepsize $h=\frac{T-t_0}{N}$, gridpoints $t_n=t_0+nh$ and off-step points $t_j^{(n)}=t_n+c_jh$ $(n\le N,\ 1\le j\le s)$, where N is a given positive integer greater than $\lfloor T-t_0\rfloor$, and denote the approximations of $y(t_{n+1})$, $y'(t_{n+1})$, $y(t_j^{(n)})$ and $y(t_j^{(n)}-\tau(t_j^{(n)}))$ by y_{n+1}, y'_{n+1}, $y_j^{(n)}$ and $z_j^{(n)}$, respectively. Let $\tau(t_j^{(n)})=(m_{j,n}-\delta_{j,n})h$ with $m_{j,n}\in\mathbb{N}$ and $\delta_{j,n}\in[0,1)$, then

$$t_j^{(n)}-\tau(t_j^{(n)})=t_{n-m_{j,n}}+(c_j+\delta_{j,n})h.$$

Further, if $c_j+\delta_{j,n}=l_{j,n}+\theta_{j,n}$ with $l_{j,n}\in\{0,1\}$ and $\theta_{j,n}\in[0,1)$, then

$$t_j^{(n)}-\tau(t_j^{(n)})=t_{n-m_{j,n}+l_{j,n}}+\theta_{j,n}h.$$

Based on these settings and the standard IRKN methods (see e.g., [10–12]), a class of adapted IRKN methods can be derived as follows:

$$\begin{cases} y_i^{(n)}=y_n+hc_iy'_n+h^2\sum\limits_{j=1}^{s}a_{ij}f\left(t_j^{(n)},y_j^{(n)},z_j^{(n)}\right), & 0\le n\le N-1,\quad 1\le i\le s,\\ y_{n+1}=y_n+hy'_n+h^2\sum\limits_{i=1}^{s}b_if\left(t_i^{(n)},y_i^{(n)},z_i^{(n)}\right), & 0\le n\le N-1,\\ y'_{n+1}=y'_n+h\sum\limits_{i=1}^{s}\hat{b}_if\left(t_i^{(n)},y_i^{(n)},z_i^{(n)}\right), & 0\le n\le N-1, \end{cases} \tag{2.4}$$

where a_{ij}, b_i, $\hat{b}_i$, c_j, $(0\le c_j\le 1)$ are some real coefficients of the methods and $z_j^{(n)}$ is approximated by the $(\mu+\nu+1)$-order Lagrange interpolation (cf. [13–16]):

$$z_j^{(n)}=\sum_{i=-\mu}^{\nu}\mathcal{L}_i(\theta_{j,n})y_{\sigma_{i,j}^n},\quad 1\le j\le s,\quad 0\le n\le N-1, \tag{2.5}$$

in which μ, ν are two nonnegative integers satisfying

$$-\mu<\nu<\min_{(j,n)\in\mathcal{J}}(m_{j,n}-\delta_{j,n})$$

with

$$\mathcal{J}=\{(j,n):j=1,2,\cdots,s;\ n=0,1,\cdots,N-1\},\quad \sigma_{i,j}^n=n-m_{j,n}+l_{j,n}+i,$$

and

$$\mathcal{L}_i(\theta_{j,n})=\prod_{\substack{k=-\mu\\k\neq i}}^{\nu}\frac{\theta_{j,n}-k}{i-k},\quad -\mu\le i\le\nu,\quad 1\le j\le s,\quad 0\le n\le N-1. \tag{2.6}$$

Here and after, when t_n, $t_j^{(n)}$ or $t_j^{(n)}-\tau(t_j^{(n)})\in[\tau_0,t_0]$, we always set the approximations y_n, y'_n, $y_j^{(n)}$, $z_j^{(n)}$ equal to their exact values $\varphi(t_n)$, $\varphi'(t_n)$, $\varphi(t_j^{(n)})$, $\varphi(t_j^{(n)}-\tau(t_j^{(n)}))$, respectively.

Let I_d and $\otimes$ be the $d\times d$ identity matrix and Kronecker product, respectively, and

$$\begin{aligned}
&e_s=(1,1,\cdots,1)^T\in\mathbb{R}^s,\quad Y_n=\left(y_1^{(n)^T},y_2^{(n)^T},\cdots,y_s^{(n)^T}\right)^T,\quad Z_n=\left(z_1^{(n)^T},z_2^{(n)^T},\cdots,z_s^{(n)^T}\right)^T,\\
&F(t_n,\widehat{Y}_n,\widehat{Z}_n)=\left(f\left(t_1^{(n)},y_1^{(n)},z_1^{(n)}\right)^T,f\left(t_2^{(n)},y_2^{(n)},z_2^{(n)}\right)^T,\cdots,f\left(t_s^{(n)},y_s^{(n)},z_s^{(n)}\right)^T\right)^T,\\
&A=(a_{ij})\in\mathbb{R}^{s\times s},\quad b=(b_1,b_2,\cdots,b_s)^T,\quad \hat{b}=(\hat{b}_1,\hat{b}_2,\cdots,\hat{b}_s)^T,\quad c=(c_1,c_2,\cdots,c_s)^T.
\end{aligned}$$

Then scheme (2.4) can be written in a more compact form:

$$\begin{cases}
Y_n=(e_s\otimes I_d)y_n+h(c\otimes I_d)y'_n+h^2(A\otimes I_d)F(t_n,Y_n,Z_n), & 0\le n\le N-1,\\
y_{n+1}=y_n+hy'_n+h^2(b^T\otimes I_d)F(t_n,Y_n,Z_n), & 0\le n\le N-1,\\
y'_{n+1}=y'_n+h(\hat{b}^T\otimes I_d)F(t_n,Y_n,Z_n), & 0\le n\le N-1.
\end{cases} \tag{2.7}$$

3 Error analysis of IRKN methods

This section will focus on error analysis of IRKN methods. For this, we first introduce the concept of consistency order. An IRKN method (2.4)-(2.5) is called consistent of order p if there exist constants $\gamma,h_0>0$ such that

$$\|y(t_{n+1})-\hat{y}_{n+1}\|_\infty\le\gamma h^{p+1},\quad \|y'(t_{n+1})-\hat{y}'_{n+1}\|_\infty\le\gamma h^{p+1},\quad \forall h\in(0,h_0], \tag{3.1}$$

where $\hat{y}_{n+1}$ and $\hat{y}'_{n+1}$ are determined by

$$\begin{cases}
\widehat{Y}_n=(e_s\otimes I_d)y(t_n)+h(c\otimes I_d)y'(t_n)+h^2(A\otimes I_d)F(t_n,\widehat{Y}_n,\widehat{Z}_n), & 0\le n\le N-1,\\
\hat{y}_{n+1}=y(t_n)+hy'(t_n)+h^2(b^T\otimes I_d)F(t_n,\widehat{Y}_n,\widehat{Z}_n), & 0\le n\le N-1,\\
\hat{y}'_{n+1}=y'(t_n)+h(\hat{b}^T\otimes I_d)F(t_n,\widehat{Y}_n,\widehat{Z}_n), & 0\le n\le N-1,
\end{cases} \tag{3.2}$$

in which

$$\widehat{Y}_n=(\hat{y}_1^{(n)^T},\hat{y}_2^{(n)^T},\cdots,\hat{y}_s^{(n)^T})^T,\quad \widehat{Z}_n=(\hat{z}_1^{(n)^T},\hat{z}_2^{(n)^T},\cdots,\hat{z}_s^{(n)^T})^T \quad \text{and} \quad \hat{z}_j^{(n)}=y(t_j^{(n)}-\tau(t_j^{(n)})).$$

Besides the concept of consistency order, the following lemmas will also be important for our analysis.

Lemma 3.1 ([16]). *Under condition* (2.3)*, the Lagrange interpolation satisfies the following estimate:*

$$\left\|\sum_{i=-\mu}^{\nu}\mathcal{L}_i(\theta_{j,n})y\left(t_{\sigma_{i,j}^n}\right)-y\left(t_j^{(n)}-\tau\left(t_j^{(n)}\right)\right)\right\|_\infty \\ \le M_{\mu+\nu+1}h^{\mu+\nu+1},\quad 1\le j\le s,\quad 0\le n\le N-1,$$

where $M_{\mu+\nu+1}>0$ is a constant indicated in (2.3).

Lemma 3.2 (Discrete Grönwall inequality [17]). *Let x_n and β_n be two real scalar sequences and $\alpha>0$ a given constant. If $x_{n+1}\le \alpha x_n+\beta_n$ for all $n\ge 0$, then*

$$x_n\le \alpha^n\left(x_0+\sum_{j=1}^{n}\alpha^{-j}\beta_{j-1}\right),\quad \forall n\ge 1.$$

Based on the above arguments, an error estimate of IRKN methods can be derived as follows.

Theorem 3.1. *Assume that conditions* (2.2)-(2.3) *are fulfilled and IRKN method* (2.4)-(2.5) *is consistent of order p. Then this method is convergent of order* $\min\{p,\mu+\nu+1\}$*, namely, there exist constant $\hat{h}>0$ and nonnegative bounded function $c(t)$ on $[t_0,T]$ such that*

$$\|y(t_n)-y_n\|_\infty\le c(t_n)h^{\min\{p,\mu+\nu+1\}},\quad 1\le n\le N,\quad h\in(0,\hat{h}]. \tag{3.3}$$

Proof. Write

$$\varepsilon_n=y(t_n)-y_n,\quad \varepsilon_n'=y'(t_n)-y_n',\quad \Delta F_n=F(t_n,\widehat{Y}_n,\widehat{Z}_n)-F(t_n,Y_n,Z_n).$$

Subtracting (2.7) from (3.2) yields that

$$\begin{cases} \widehat{Y}_n-Y_n=(e_s\otimes I_d)\varepsilon_n+h(c\otimes I_d)\varepsilon_n'+h^2(A\otimes I_d)\Delta F_n, & 0\le n\le N-1,\\ \varepsilon_{n+1}=\varepsilon_n+h\varepsilon_n'+h^2(b^T\otimes I_d)\Delta F_n+y(t_{n+1})-\hat{y}_{n+1}, & 0\le n\le N-1,\\ \varepsilon_{n+1}'=\varepsilon_n'+h(\hat{b}^T\otimes I_d)\Delta F_n+y'(t_{n+1})-\hat{y}_{n+1}', & 0\le n\le N-1. \end{cases} \tag{3.4}$$

Taking l_∞-norm on the both sides of each equation in (3.4) and then applying some common properties of l_∞-norm and Kronecker product infer for $0\le n\le N-1$ and $h\in(0,h_0]$ that

$$\|\widehat{Y}_n-Y_n\|_\infty\le\|\varepsilon_n\|_\infty+h\|\varepsilon_n'\|_\infty+h^2\|A\|_\infty\|\Delta F_n\|_\infty, \tag{3.5a}$$

$$\|\varepsilon_{n+1}\|_\infty\le\|\varepsilon_n\|_\infty+h\|\varepsilon_n'\|_\infty+h^2\|b^T\|_\infty\|\Delta F_n\|_\infty+\gamma h^{p+1}, \tag{3.5b}$$

$$\|\varepsilon_{n+1}'\|_\infty\le\|\varepsilon_n'\|_\infty+h\|\hat{b}^T\|_\infty\|\Delta F_n\|_\infty+\gamma h^{p+1}, \tag{3.5c}$$

where condition $\|c\|_\infty \le 1$ and p-order consistency conditions (3.1) have been used. Let

$$L_0 = \max_{-\mu\le i\le \nu} \sup_{\theta\in[0,1)} |\mathcal{L}_i(\theta)|.$$

With (2.5) and Lemma 3.1, we have for $0\le n\le N-1$ that

$$\begin{aligned}
\|\widehat{Z}_n - Z_n\|_\infty &= \max_{1\le j\le s}\left\| \sum_{i=-\mu}^{\nu} \mathcal{L}_i(\theta_{j,n}) y_{\sigma^n_{i,j}} - y\left(t_j^{(n)} - \tau\left(t_j^{(n)}\right)\right)\right\|_\infty \\
&\le \max_{1\le j\le s}\left\| \sum_{i=-\mu}^{\nu} \mathcal{L}_i(\theta_{j,n}) \left[y_{\sigma^n_{i,j}} - y(t_{\sigma^n_{i,j}})\right]\right\|_\infty \\
&\quad + \max_{1\le j\le s}\left\| \sum_{i=-\mu}^{\nu} \mathcal{L}_i(\theta_{j,n}) y(t_{\sigma^n_{i,j}}) - y\left(t_j^{(n)} - \tau\left(t_j^{(n)}\right)\right)\right\|_\infty \\
&\le L_0 \sum_{i=-\mu}^{\nu} \max_{1\le j\le s} \|y_{\sigma^n_{i,j}} - y(t_{\sigma^n_{i,j}})\|_\infty + M_{\mu+\nu+1} h^{\mu+\nu+1} \\
&\le L_0(\mu+\nu+1) \max_{0\le k\le n} \|\varepsilon_k\|_\infty + M_{\mu+\nu+1} h^{\mu+\nu+1}. \qquad (3.6)
\end{aligned}$$

This, together with Lipschitz condition (2.2), implies for $0\le n\le N-1$ that

$$\begin{aligned}
\|\Delta F_n\|_\infty &\le L_1\|\widehat{Y}_n - Y_n\|_\infty + L_2\|\widehat{Z}_n - Z_n\|_\infty \\
&\le L_1\|\widehat{Y}_n - Y_n\|_\infty + L_0 L_2(\mu+\nu+1) \max_{0\le k\le n}\|\varepsilon_k\|_\infty + L_2 M_{\mu+\nu+1} h^{\mu+\nu+1}. \qquad (3.7)
\end{aligned}$$

Let $h_1 > 0$ be a constant such that $0 < h^2 L_1 \|A\|_\infty < 1$ for all $h\in(0,h_1]$, $h_2 = \min\{h_0, h_1\}$, $L_3 = \frac{1}{1-h_2^2 L_1\|A\|_\infty}$ and $L_4 = L_0(\mu+\nu+1)$. Substituting (3.5a) into (3.7) follows for $0\le n\le N-1$ and $h\in(0,h_2]$ that

$$\|\Delta F_n\|_\infty \le L_3\left(L_1\|\varepsilon_n\|_\infty + hL_1\|\varepsilon'_n\|_\infty + L_2 L_4 \max_{0\le k\le n}\|\varepsilon_k\|_\infty + L_2 M_{\mu+\nu+1} h^{\mu+\nu+1}\right). \qquad (3.8)$$

Combining (3.5b) with (3.8) derives that

$$\begin{aligned}
\|\varepsilon_{n+1}\|_\infty \le& \left[1 + h^2\|b^T\|_\infty L_3(L_1 + L_2L_4)\right] \max_{0\le k\le n}\|\varepsilon_k\|_\infty + h\left(1 + h^2\|b^T\|_\infty L_1 L_3\right) \max_{0\le k\le n}\|\varepsilon'_k\|_\infty \\
&+ \|b^T\|_\infty L_2 L_3 M_{\mu+\nu+1} h^{\mu+\nu+3} + \gamma h^{p+1}, \quad 0\le n\le N-1, \quad h\in(0,h_2]. \qquad (3.9)
\end{aligned}$$

Inserting (3.8) into (3.5c) gives that

$$\begin{aligned}
\|\varepsilon'_{n+1}\|_\infty \le& \left(1 + h^2\|\hat{b}^T\|_\infty L_1 L_3\right)\|\varepsilon'_n\|_\infty + h\|\hat{b}^T\|_\infty L_3(L_1 + L_2L_4) \max_{0\le k\le n}\|\varepsilon_k\|_\infty \\
&+ \|\hat{b}^T\|_\infty L_2 L_3 M_{\mu+\nu+1} h^{\mu+\nu+2} + \gamma h^{p+1}, \quad 0\le n\le N-1, \quad h\in(0,h_2]. \qquad (3.10)
\end{aligned}$$

Define

$$\mathcal{E}_n=\left(\max_{0\le k\le n}\|\varepsilon_k\|_\infty,\max_{0\le k\le n}\|\varepsilon'_k\|_\infty\right)^T,\quad B_h=L_2L_3M_{\mu+\nu+1}\left(h\|b^T\|_\infty,\|\hat{b}^T\|_\infty\right)^T,$$

and

$$P_h=\begin{pmatrix}1+h^2\|b^T\|_\infty L_3(L_1+L_2L_4) & h\left(1+h^2\|b^T\|_\infty L_1L_3\right)\\ h\|\hat{b}^T\|_\infty L_3(L_1+L_2L_4) & 1+h^2\|\hat{b}^T\|_\infty L_1L_3\end{pmatrix}.$$

A direct computation shows that

$$\|B_h\|_\infty=L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty\quad\text{and}\quad\|P_h\|_\infty=1+\mathcal{O}(h)$$

as $h\to0^+$. This means that there exist constants $\rho_0,h_3>0$ such that

$$\|B_h\|_\infty=L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty,\quad\|P_h\|_\infty\le1+\rho_0h,\quad h\in(0,h_3].\tag{3.11}$$

Moreover, by (3.9) and (3.10), it holds that

$$\|\mathcal{E}_{n+1}\|_\infty\le\|P_h\|_\infty\|\mathcal{E}_n\|_\infty+\|B_h\|_\infty h^{\mu+\nu+2}+\gamma h^{p+1},\quad0\le n\le N-1,\quad h\in(0,h_2].\tag{3.12}$$

Applying Lemma 3.2 to (3.12) and using condition: $\|\mathcal{E}_0\|_\infty=0$ derive that

$$\|\mathcal{E}_n\|_\infty\le\sum_{j=1}^n\|P_h\|_\infty^{n-j}\left(\|B_h\|_\infty h^{\mu+\nu+2}+\gamma h^{p+1}\right),\quad1\le n\le N,\quad h\in(0,h_2].\tag{3.13}$$

Let $\hat{h}=\min\{h_2,h_3,1\}$. Then, when $1\le n\le N$ and $h\in(0,\hat{h}]$, a combination of (3.11), (3.13), inequality: $1+x\le\exp(x)$ $(x\ge0)$ and equality: $nh=t_n-t_0$ generates that

$$\begin{aligned}\|\mathcal{E}_n\|_\infty&\le\sum_{j=1}^n(1+\rho_0h)^{n-j}\left(L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty h^{\mu+\nu+2}+\gamma h^{p+1}\right)\\&\le(1+\rho_0h)^n nh\left(L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty h^{\mu+\nu+1}+\gamma h^p\right)\\&\le\exp[\rho_0(t_n-t_0)](t_n-t_0)\left(L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty+\gamma\right)h^{\min\{p,\mu+\nu+1\}}.\end{aligned}\tag{3.14}$$

This, together with the fact:

$$\|\varepsilon_n\|_\infty\le\max_{0\le k\le n}\|\varepsilon_k\|_\infty\le\|\mathcal{E}_n\|_\infty\quad\text{for}\quad1\le n\le N,$$

concludes error estimate (3.3) with

$$c(t_n)=\exp[\rho_0(t_n-t_0)](t_n-t_0)\left(L_2L_3M_{\mu+\nu+1}\|\hat{b}^T\|_\infty+\gamma\right).$$

Therefore the theorem is proved. □

4 Global stability of IRKN methods

In this section, we present an analysis to the global stability of IRKN methods (2.4)-(2.5). For this purpose, besides problems (2.1), we also need to consider their corresponding perturbation problems with different initial function $\psi(t)$:

$$\tilde{y}''(t)=f(t,\tilde{y}(t),\tilde{y}(t-\tau(t))), \qquad t\in[t_0,T]; \tag{4.1a}$$
$$\tilde{y}(t)=\psi(t), \quad \tilde{y}'(t)=\psi'(t), \qquad t\in[\tau_0,t_0]. \tag{4.1b}$$

When an IRKN method (2.4)-(2.5) is applied to problems (4.1), we write the approximations of $\tilde{y}(t_n)$, $\tilde{y}'(t_n)$, $\tilde{y}(t_j^{(n)})$ and $\tilde{y}(t_j^{(n)}-\tau(t_j^{(n)}))$ as $\tilde{y}_n$, $\tilde{y}'_n$, $\tilde{y}_j^{(n)}$ and $\tilde{z}_j^{(n)}$, respectively. An IRKN method (2.7) is called globally stable if there exists constants $\mathcal{H},\tilde{h}>0$ such that

$$\|y_n-\tilde{y}_n\|_\infty\le\mathcal{H}\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty, \quad 1\le n\le N, \quad h\in(0,\tilde{h}].$$

A global stability criterion of IRKN methods (2.4)-(2.5) can be stated as follows.

Theorem 4.1. *Assume that Lipschitz condition* (2.2) *holds. Then IRKN method* (2.4)-(2.5) *is globally stable.*

Proof. Write

$$\eta_n=y_n-\tilde{y}_n, \qquad \eta'_n=y'_n-\tilde{y}'_n,$$
$$\widetilde{Y}_n=\left(\tilde{y}_1^{(n)^T},\tilde{y}_2^{(n)^T},\cdots,\tilde{y}_s^{(n)^T}\right)^T, \qquad \widetilde{Z}_n=\left(\tilde{z}_1^{(n)^T},\tilde{z}_2^{(n)^T},\cdots,\tilde{z}_s^{(n)^T}\right)^T,$$
$$\hat{\Delta}F_n=F(t_n,Y_n,Z_n)-F(t_n,\widetilde{Y}_n,\widetilde{Z}_n).$$

With these notations and (2.7), we have that

$$\begin{cases} Y_n-\widetilde{Y}_n=(e_s\otimes I_d)\eta_n+h(c\otimes I_d)\eta'_n+h^2(A\otimes I_d)\hat{\Delta}F_n, & 0\le n\le N-1,\\ \eta_{n+1}=\eta_n+h\eta'_n+h^2(b^T\otimes I_d)\hat{\Delta}F_n, & 0\le n\le N-1,\\ \eta'_{n+1}=\eta'_n+h(\hat{b}^T\otimes I_d)\hat{\Delta}F_n, & 0\le n\le N-1. \end{cases} \tag{4.2}$$

It follows from (4.2), $\|c\|_\infty\le 1$ and the common properties of l_∞-norm and Kronecker product that

$$\|Y_n-\widetilde{Y}_n\|_\infty\le\|\eta_n\|_\infty+h\|\eta'_n\|_\infty+h^2\|A\|_\infty\|\hat{\Delta}F_n\|_\infty, \qquad 0\le n\le N-1, \tag{4.3a}$$
$$\|\eta_{n+1}\|_\infty\le\|\eta_n\|_\infty+h\|\eta'_n\|_\infty+h^2\|b^T\|_\infty\|\hat{\Delta}F_n\|_\infty, \qquad 0\le n\le N-1, \tag{4.3b}$$
$$\|\eta'_{n+1}\|_\infty\le\|\eta'_n\|_\infty+h\|\hat{b}^T\|_\infty\|\hat{\Delta}F_n\|_\infty, \qquad 0\le n\le N-1. \tag{4.3c}$$

Based on (2.5), it can be inferred that

$$\begin{aligned}\|Z_n-\widetilde{Z}_n\|_\infty &\le \max_{1\le j\le s}\sum_{i=-\mu}^{\nu}\|\mathcal{L}_i(\theta_{j,n})\|_\infty\|y_{\sigma_{i,j}^n}-\tilde{y}_{\sigma_{i,j}^n}\|_\infty\\ &\le L_0(\mu+\nu+1)\max\Big\{\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\max_{0\le k\le n}\|\eta_k\|_\infty\Big\}\\ &\le L_0(\mu+\nu+1)\Big(\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty+\max_{0\le k\le n}\|\eta_k\|_\infty\Big),\quad 0\le n\le N-1. \end{aligned} \tag{4.4}$$

Also, by Lipschitz condition (2.2), it holds that

$$\|\hat{\Delta}F_n\|_\infty\le L_1\|Y_n-\widetilde{Y}_n\|_\infty+L_2\|Z_n-\widetilde{Z}_n\|_\infty,\quad 0\le n\le N-1. \tag{4.5}$$

Substituting (4.3a) and (4.4) into (4.5) yields for $0\le n\le N-1$ that

$$\begin{aligned}\|\hat{\Delta}F_n\|_\infty\le &L_1\left(\|\eta_n\|_\infty+h\|\eta_n'\|_\infty+h^2\|A\|_\infty\|\hat{\Delta}F_n\|_\infty\right)\\ &+L_0L_2(\mu+\nu+1)\Big(\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty+\max_{0\le k\le n}\|\eta_k\|_\infty\Big).\end{aligned} \tag{4.6}$$

Let $h_1>0$ be the constant used in the proof of Theorem 3.1 and $\tilde{L}_1=\frac{1}{1-h_1^2L_1\|A\|_\infty}$. When $h\in(0,h_1]$, we can obtain from inequality (4.6) for $0\le n\le N-1$ that

$$\begin{aligned}\|\hat{\Delta}F_n\|_\infty\le &\tilde{L}_1\Big[L_1\left(\|\eta_n\|_\infty+h\|\eta_n'\|_\infty\right)\\ &+L_0L_2(\mu+\nu+1)\Big(\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty+\max_{0\le k\le n}\|\eta_k\|_\infty\Big)\Big].\end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.3b) yields that

$$\begin{aligned}\|\eta_{n+1}\|_\infty\le &\Big[1+h^2\|b^T\|_\infty\tilde{L}_1\left(L_1+L_0L_2(\mu+\nu+1)\right)\Big]\max_{0\le k\le n}\|\eta_k\|_\infty\\ &+h(1+h^2\|b^T\|_\infty\tilde{L}_1L_1)\|\eta_n'\|_\infty+h^2\|b^T\|_\infty\tilde{L}_1L_0L_2(\mu\\ &+\nu+1)\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\quad 0\le n\le N-1,\quad h\in(0,h_1].\end{aligned} \tag{4.8}$$

A combination of (4.3c) and (4.7) generates that

$$\begin{aligned}\|\eta_{n+1}'\|_\infty\le &\Big[1+h^2\|\hat{b}^T\|_\infty\tilde{L}_1L_1\Big]\|\eta_n'\|_\infty+h\|\hat{b}^T\|_\infty\tilde{L}_1[L_1+L_0L_2(\mu+\nu+1)]\max_{0\le k\le n}\|\eta_k\|_\infty\\ &+h\|\hat{b}^T\|_\infty\tilde{L}_1L_0L_2(\mu+\nu+1)\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\quad 0\le n\le N-1,\quad h\in(0,h_1].\end{aligned} \tag{4.9}$$

Define

$$\begin{aligned}&E_n=\left(\max_{0\le k\le n}\|\eta_k\|_\infty,\max_{0\le k\le n}\|\eta_k'\|_\infty\right)^T,\quad \mathcal{B}_h=h\tilde{L}_1L_0L_2(\mu+\nu+1)(h\|b^T\|_\infty,\|\hat{b}^T\|_\infty)^T,\\ &Q_h=\begin{pmatrix}1+h^2\|b^T\|_\infty\tilde{L}_1\left(L_1+L_0L_2(\mu+\nu+1)\right) & h(1+h^2\|b^T\|_\infty\tilde{L}_1L_1)\\ h\|\hat{b}^T\|_\infty\tilde{L}_1[L_1+L_0L_2(\mu+\nu+1)] & 1+h^2\|\hat{b}^T\|_\infty\tilde{L}_1L_1\end{pmatrix}.\end{aligned}$$

A direct computation gives that $\|\mathcal{B}_h\|_\infty=\mathcal{O}(h)$ and $\|Q_h\|_\infty=1+\mathcal{O}(h)$ as $h\to 0^+$. This implies that there exist constants $\rho_1,\rho_2,\tilde{h}_1>0$ such that

$$\|\mathcal{B}_h\|_\infty\le\rho_1 h,\quad \|Q_h\|_\infty\le 1+\rho_2 h,\quad h\in(0,\tilde{h}_1]. \tag{4.10}$$

Moreover, when set $\tilde{h}=\min\{h_1,\tilde{h}_1\}$, it can be reached by (4.8) and (4.9) that

$$\|E_{n+1}\|_\infty\le\|Q_h\|_\infty\|E_n\|_\infty+\|\mathcal{B}_h\|_\infty\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\quad 0\le n\le N-1,\quad h\in(0,\tilde{h}]. \tag{4.11}$$

Applying Lemma 3.2 to (4.11) and considering condition: $\|E_0\|_\infty=0$ derive that

$$\|E_n\|_\infty\le\sum_{j=1}^{n}\|Q_h\|_\infty^{n-j}\|\mathcal{B}_h\|_\infty\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\quad 1\le n\le N,\quad h\in(0,\tilde{h}]. \tag{4.12}$$

Embedding (4.10) into (4.12) and using inequalities: $1+x\le\exp(x)$ $(x\ge 0)$ and $nh=t_n-t_0\le T-t_0$ $(1\le n\le N)$ deduce for $1\le n\le N$ and $h\in(0,\tilde{h}]$ that

$$\begin{aligned}\|E_n\|_\infty&\le\sum_{j=1}^{n}(1+\rho_2 h)^{n-j}\rho_1 h\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty\\&\le\sum_{j=1}^{n}\exp[\rho_2(n-j)h]\rho_1 h\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty\\&\le\exp[\rho_2(n-1)h]\rho_1 nh\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty\\&\le\exp[\rho_2(T-t_0)]\rho_1(T-t_0)\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty.\end{aligned} \tag{4.13}$$

Combining (4.13) and fact:

$$\|\eta_n\|_\infty\le\max_{0\le k\le n}\|\eta_k\|_\infty\le\|E_n\|_\infty,\quad (1\le n\le N),$$

concludes that

$$\|\eta_n\|_\infty\le\exp[\rho_2(T-t_0)]\rho_1(T-t_0)\max_{\tau_0\le t\le t_0}\|\varphi(t)-\psi(t)\|_\infty,\quad 1\le n\le N,\quad h\in(0,\tilde{h}]. \tag{4.14}$$

This shows that the IRKN method is globally stable. Hence the proof is completed. □

5 Application to an IBVP of nonlinear delay wave equations

In order to give a numerical verification to the computational effectiveness and accuracy of IRKN methods (2.4)-(2.5), we consider an application to the following IBVP of nonlinear delay wave equations:

$$\begin{cases}u_{tt}(x,t)=\dfrac{1}{200}u_{xx}(x,t)+\dfrac{1}{10}u(x,t-e^{-t})+\dfrac{1}{1+u^2(x,t)}+\tilde{f}(x,t), & (x,t)\in[0,\pi]\times[0,2],\\ u(x,t)=\sin(x)\exp(-2t),\quad u_t(x,t)=-2\sin(x)\exp(-2t), & (x,t)\in[0,\pi]\times[-1,0],\\ u(0,t)=0,\quad u(\pi,t)=0, & t\in[0,2],\end{cases} \tag{5.1}$$

where $\tilde{f}(x,t)$ is a given function such that problem (5.1) has the exact solution $u(x,t)=e^{-2t}\sin(x)$. For numerically solving problem (5.1), in what follows, we will first use a fourth-order compact difference scheme (see e.g., [18]) to discretize the spatial variable in (5.1) and then solve the derived second-order delay-initial-value problem by IRKN methods (2.4)-(2.5).

According to reference [18], one has the following fourth-order compact difference scheme with spatial stepsize $\Delta x=\frac{\pi}{k}$ $(k\geq 4)$ and mesh points $x_j=j\Delta x$ $(1\leq j\leq k)$ for approximating $u_{xx}(x_j,t)$:

$$u_{xx}(x_j,t)\approx\left(1+\frac{\Delta x^2}{12}\delta_x^2\right)^{-1}\delta_x^2u_j(t),\quad j=1,2,\cdots,k-1, \tag{5.2}$$

where

$$u_j(t)\approx u(x_j,t)\quad \text{and}\quad \delta_x^2u_j(t)=\frac{u_{j-1}(t)-2u_j(t)+u_{j+1}(t)}{\Delta x^2}.$$

Let

$$\begin{aligned}
&y(t)=(u_1(t),u_2(t),\cdots,u_{k-1}(t))^T,\quad \hat{L}_1=\text{tridiag}\left\{\frac{1}{12},\frac{5}{6},\frac{1}{12}\right\},\quad \hat{L}_2=\frac{\text{tridiag}\{1,-2,1\}}{200(\Delta x)^2},\\
&L=\hat{L}_1^{-1}\hat{L}_2,\quad \hat{f}(y(t))=\left(\frac{1}{1+u_1^2(t)},\frac{1}{1+u_2^2(t)},\cdots,\frac{1}{1+u_{k-1}^2(t)}\right)^T,\\
&\tilde{f}(t)=(\tilde{f}(x_1,t),\tilde{f}(x_2,t),\cdots,\tilde{f}(x_{k-1},t))^T,\quad \varphi(t)=e^{-2t}(\sin(x_1),\sin(x_2),\cdots,\sin(x_{k-1}))^T,\\
&g(t,y(t),y(t-e^{-t}))=\frac{1}{10}y(t-e^{-t})+\hat{f}(y(t))+\tilde{f}(t).
\end{aligned}$$

Then, with the above notations and (5.2), problem (5.1) can be discretized as the following second-order delay-initial-value problem:

$$y''(t)=Ly(t)+g(t,y(t),y(t-e^{-t})),\qquad t\in[0,2]; \tag{5.3a}$$
$$y(t)=\varphi(t),\quad y'(t)=\varphi'(t),\qquad t\in[-1,0]. \tag{5.3b}$$

It can be checked that the right-function $f(t,y,z):=Ly+g(t,y,z)$ of problem (5.3) satisfies Lipschitz condition (2.2) with $L_1=1+\frac{0.03}{(\Delta x)^2}$ and $L_2=0.1$. This shows that problem (5.3) is stiff with respect to the second variable y of $f(t,y,z)$ and its stiffness becomes stronger with the decrease of Δx. Moreover, it follows from Theorem 3.1 and Theorem 4.1 that IRKN methods (2.4)-(2.5) of order p are convergent of order $\min\{p,\mu+\nu+1\}$ and globally stable for problem (5.3).

In the following, we present a numerical illustration to the above theoretical results. For convenience, we write an IRKN method (2.4)-(2.5) with p-order consistency and q-order interpolation as *IRKN*(p,q), where $q=\mu+\nu+1$, and use the following formulaes to compute the global errors $err(h)$ and convergence orders $p(h)$ of IRKN methods with

Table 1: Global errors of IRKN(p,q) $(p,q$=2,3,4) for problem (5.3).

h	IRKN(2,2)	IRKN(3,2)	IRKN(4,2)	IRKN(2,3)	IRKN(3,3)	IRKN(4,3)	IRKN(2,4)	IRKN(3,4)	IRKN(4,4)
1/10	2.0282e-02	7.1401e-04	7.3003e-04	1.9786e-02	8.2095e-06	1.5288e-05	1.9773e-02	8.0025e-06	9.4535e-07
1/20	5.1122e-03	1.6464e-04	1.7093e-04	4.9411e-03	8.4806e-07	1.7257e-06	4.9392e-03	9.3155e-07	5.9548e-08
1/40	1.2765e-03	4.2518e-05	4.4425e-05	1.2348e-03	9.6404e-08	2.0582e-07	1.2346e-03	1.1357e-07	3.7154e-09
1/80	3.1943e-04	1.0188e-05	1.0389e-05	3.0865e-04	1.0201e-08	2.3909e-08	3.0863e-04	1.4013e-08	2.3326e-10
1/160	7.9433e-05	2.2860e-06	2.3401e-06	7.7159e-05	1.3245e-09	3.0714e-09	7.7156e-05	1.7453e-09	1.4368e-11

Table 2: Convergence orders of IRKN(p,q) $(p,q$=2,3,4) for problem (5.3).

h	IRKN(2,2)	IRKN(3,2)	IRKN(4,2)	IRKN(2,3)	IRKN(3,3)	IRKN(4,3)	IRKN(2,4)	IRKN(3,4)	IRKN(4,4)
1/10	–	–	–	–	–	–	–	–	–
1/20	1.9881	2.1166	2.0945	2.0016	3.2750	3.1471	2.0012	3.1027	3.9887
1/40	2.0018	1.9532	1.9440	2.0006	3.1370	3.0677	2.0003	3.0361	4.0025
1/80	1.9986	2.0612	2.0964	2.0002	3.2405	3.1057	2.0001	3.0188	3.9935
1/160	2.0077	2.1560	2.1503	2.0001	2.9451	2.9606	2.0000	3.0052	4.0210

stepsize h:

$$err(h)=\max_{0\le n\le N}\|y(t_n)-y_n\|_\infty,\quad p(h)=\log_2\left[\frac{err(2h)}{err(h)}\right].$$

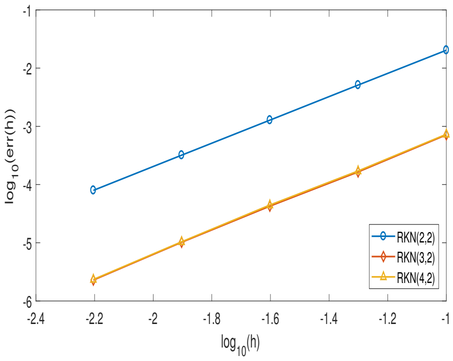
The subsequent numerical experiments will based on nine kinds of implicit IRKN(p,q) $(p,q=2,3,4)$, where the coefficients of the corresponding p-order main schemes (2.4) can be obtained by the order conditions in Shi, Zhang & Wang [12] and the interpolation coefficients in (2.5) can be determined by setting $(\mu,\nu)=(1,0),(1,1),(2,1)$ in formula (2.6), respectively. Taking spatial stepsize $\Delta x=\frac{\pi}{200}$ and temporal stepsizes $h=\frac{2}{N}(N=20,40,80,160,320)$, and then applying the above nine kinds of IRKN(p,q) $(p,q=2,3,4)$ to solve problem (5.3), respectively, the generated global errors and convergence orders are displayed respectively in Table 1 and Table 2. Moreover, in Fig. 1, we also plot the global errors of IRKN$(p,q)(p,q=2,3,4)$ versus stepsizes h, which are plotted in Log-Log scale. These numerical results further confirm the computational effectiveness of IRKN methods and the theoretical accuracy shown in Theorem 3.1.

Next, we give an insight into the global stability of IRKN methods (2.4)-(2.5). For this, besides problem (5.3), we also consider its perturbation problem with initial perturbation $\epsilon\sin(t)\circ e_k$:

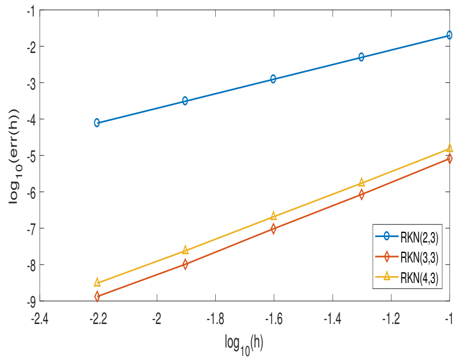
$$\tilde{y}''(t)=L\tilde{y}(t)+g(t,\tilde{y}(t),\tilde{y}(t-e^{-t})),\qquad t\in[0,2]; \tag{5.4a}$$
$$\tilde{y}(t)=\psi(t),\quad \tilde{y}'(t)=\psi'(t),\qquad t\in[-1,0], \tag{5.4b}$$

where $0<\epsilon\le 1$, $e_k=(1,1,\cdots,1)^{\rm T}\in\mathbb{R}^{k-1}$, $\psi(t)=\varphi(t)+\epsilon\sin(t)\circ e_k$ and $\circ$ is the Schur product. A simple computational gives that

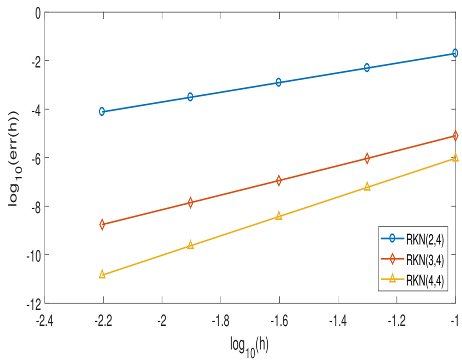
$$\max_{-1\le t\le 0}\|\varphi(t)-\psi(t)\|_\infty=\epsilon\sin(1).$$



(a) Global errors of IRKN(p,2) (p=2,3,4)



(b) Global errors of IRKN(p,3) (p=2,3,4)



(c) Global errors of IRKN(p,4) (p=2,3,4)

Figure 1: Global errors of IRKN(p,q) (q=2,3,4) against h in Log-Log scale for problem (5.3).

Let

$$\mathcal{M}(\epsilon,t_n)=\begin{cases} \|y_n-\tilde{y}_n\|_\infty/\epsilon\sin(1), & 0<\epsilon\le 1, \\ 0, & \epsilon=0, \end{cases} \qquad 1\le n\le N:=\frac{2}{h},$$

where y_n and $\tilde{y}_n$ are the numerical solutions of problems (5.3) and (5.4) solved by an IRKN method with stepsize h, respectively. As an example, applying IRKN(4,4) with $h=\frac{1}{160}$ to solve problems (5.3) and (5.4), we can derive that

$$\max_{0\le\epsilon\le 1}\max_{1\le n\le 320}\mathcal{M}(\epsilon,t_n)\approx 2.2367,$$

which implies that IRKN(4,4) with $h=\frac{1}{160}$ for problem (5.3) is globally stable with stability constant $\mathcal{H}\approx 2.2367$. For the other IRKN methods, we can conclude from the above similar approach that they are all globally stable for problems (5.3). This further illustrates the global stability result stated in Theorem 4.1.

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