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Improvements to Enhance Robustness of Third-Order Scale-Independent WENO-Z Schemes

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Abstract. Although there are many improvements to WENO3-Z with the target to achieve the optimal order in the occurrence of the first-order critical point (CP_1), they mainly address resolution performance, while the robustness of schemes is of less concern and lacks understanding accordingly. In light of our analysis considering the occurrence of critical points within grid intervals, we theoretically prove that it is impossible for a scale-independent scheme that has the stencil of WENO3-Z to fulfill the above order achievement, and current scale-dependent improvements barely fulfill the job when CP_1 occurs at the middle of the grid cell. In order to achieve scale-independent improvements, we devise new smoothness indicators that increase the error order from 2 to 4 when CP_1 occurs and perform more stably. Meanwhile, we construct a new global smoothness indicator that increases the error order from 4 to 5 similarly, through which new nonlinear weights with regard to WENO3-Z are derived and new scale-independent improvements, namely WENO-Z$_{\text{ES2}}$ and -Z$_{\text{ES3}}$, are acquired. Through 1D scalar and Euler tests, as well as 2D computations, in comparison with typical scale-dependent improvements, the following performances of the proposed schemes are demonstrated: the schemes can achieve third-order accuracy at CP_1 no matter its location in the stencil, indicate high resolution in resolving flow subtleties, and manifest strong robustness in hypersonic simulations (e.g., the accomplishment of computations on hypersonic half-cylinder flow with Mach numbers reaching 16 and 19, respectively, as well as essentially non-oscillatory solutions of inviscid sharp double cone flow at $M=9.59$), which contrasts the comparative WENO3-Z improvements.

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1 Introduction

It is well known that the weighted essentially non-oscillatory (WENO) implementation in [1], abbreviated as WENO-JS, has been widely applied in computational fluid dynamics. The third-order version, WENO3-JS, is particularly interesting for engineers for its efficiency and robustness. Therefore, the specific improvements of WENO-JS from the view of WENO-Z [2–4] are considered in this study.

The original third-order WENO-Z, or WENO3-Z, was derived [4] by following the implementation of the fifth-order WENO-Z [2]. Reference [5] indicated that WENO3-Z would not satisfy the accuracy relation to achieve the third-order accuracy, especially in the occurrence of critical points. Several improvements were proposed to recover the optimal order in the case of first-order critical points ($f'=0$, $f''\&f'''\neq 0$) under the framework of WENO3-Z, such as WENO-NP3 [6], -F3 [7], -NN3 [8], and -PZ3 [9]. As indicated in [10], although the improvements were proposed from different perspectives, the global smoothness indicators (τ) therein except τ_3 actually assumed essentially the same form, namely, $c(f_{(i-1)}-2f_i+f_{(i+1)})^2$, where f denotes the variable and $c>0$ is the coefficient. Moreover, Reference [11] pointed out the following: (1) WENO-NP3, -F3, and -NN3 cannot fulfill the order recovery in L_∞-norm as long as the critical point occurs at the half nodes, while WENO-NN3 even fails to achieve the order once the critical point appears; (2) In order to achieve the order recovery, the constructions of the above schemes rely on the assumption that the first-order critical point occurs at x_j regarding the discretization of $(\partial f/\partial x)_j$. However, this assumption is not comprehensive and would yield the incorrect formulation in the case of WENO-NN3; (3) All the above-mentioned improvement schemes are scale-dependent, meaning that inconsistent results will be yielded when a different variable scale or length scale is employed.

As we once indicated [12], the critical point could occur at any position within the grid cell, and consequent accuracy relations of smoothness indicators might be varied. Reference [11] further elaborated the indication and systemized the analysis considering the occurrence of critical points within grid intervals. According to the analysis in [11], we derived the correct solution for WENO-NN3, which was validated by computation. Based on the theoretical outcomes, two scale-independent third-order schemes were devised. Both schemes expand the grid stencil(s) of WENO3-Z, and the one that only expands the downwind stencil makes use of the mapping in [12] and is referred to as WENO3-ZM. In [11], WENO3-ZM seemed more appealing because it fulfilled all tests there. Both schemes succeeded in optimal L_∞-order recovery at the first-order critical point; however, in our subsequent tests where the inflow had a large Mach number, we found WENO3-ZM indicates inadequate robustness (e.g., oscillations or even blow-up occurs in supersonic cylinder flows when the Mach number is larger than 4). Evidently, such issues should be analyzed and solutions be found to improve robustness, which is critical to practical applications.