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Lattice Boltzmann Model for a Class of Viscous Wave Equation

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Abstract. In this work, a new lattice Boltzmann model for a class of viscous wave equation is proposed through the variable transformation, which eliminates the mixed third order partial derivative term of time and space. Some numerical tests are performed to validate the present model, and the results show that the present model has a second-order convergence rate in space.

AMS subject classifications: 76M28, 65M12

Key words: Lattice Boltzmann model, a class of viscous wave equation, nerve conduction equation, Chapman-Enskog analysis, second-order accuracy.

1 Introduction

A class of viscous wave equation with initial-boundary value condition can be written as

$$\begin{cases} \partial_{tt}u-\alpha\nabla^2\partial_t u-\gamma\nabla^2 u=S(u(\mathbf{x},t),\partial_t u(\mathbf{x},t),\mathbf{x},t), & \mathbf{x}\in\Omega, \\ u(\mathbf{x},t)=0, & \mathbf{x}\in\partial\Omega, \\ \partial_t u(\mathbf{x},0)=v_0(\mathbf{x}), \quad u(\mathbf{x},0)=u_0(\mathbf{x}), & \mathbf{x}\in\Omega, \end{cases} \tag{1.1}$$

where ∇ is the gradient operator with respect to the position $\mathbf{x}$ in n $(n=1,2,3,\cdots)$ dimensions, $\partial\Omega$ is the boundary of the computational domain Ω. $\alpha>0$ and $\gamma\geq 0$ are constant

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parameters, S is the known function of variables u, $\partial_t u$, $\mathbf{x}$ and t. Eq. (1.1) can be degenerated into some special equations with different parameters, such as nerve conduction equation with $\alpha=\gamma=1$, $S=p(u)\partial_t u+g(u)+h(\mathbf{x},t)$, Stokes' wave equation [1, 2] with $\alpha=4\mu/3\rho$, $\gamma=c_0^2$, $S=-c_0^2 h(\mathbf{x},t)$ and so on [3, 4]. With the development of the technology, some numerical methods, including finite element method [5], finite difference method [6], multi-step splitting method [7], Block preconditioning strategies [8], and domain decomposition method [9] are used to solve the viscous wave equation.

Compared with above mentioned methods, the lattice Boltzmann (LB) method, as a mesoscopic numerical approach, not only gains a great success in the study of the complex fluid flow [10–13], but also attains increasing attention in solving nonlinear partial differential equation (PDE), including the convection-diffusion equation (CDE) [14–17], Kardar-Parisi-Zhang (KPZ) equation [18], a class of the third order PDE [19], and sixth order PDE [20]. However, most of existing LB models are suitable for the CDE or one-dimensional high order PDE, and can not be used to solve high-dimensional high order PDE. Furthermore, previous LB models for one-dimensional high order PDE need more discrete velocity directions and special boundary treatment. Overcome the drawback of above LB models for high order PDE mentioned, a new lattice Boltzmann model for a class of viscous wave equation is proposed through the variable transformation, where the PDE with the mixed third order partial derivative term of time and space can be transformed into special coupled diffusion equations. And then using the LB model to solve the coupled equations. This idea can be used to solve other special high order PDE.

The rest of the paper is organized as follows. In Section 2, a lattice Bhatnagar-Gross-Krook (LBGK) model for a class of viscous wave equation is presented. In Section 3, some numerical simulations are performed to test the present model, and finally, a brief summary is given in Section 4.

2 LBGK model

Eq. (1.1) can be written in the following form

$$\begin{cases} \partial_t v=\nabla^2(\gamma u+\alpha v)+s, & \\ \partial_t u=v, & \mathbf{x}\in\Omega, \\ u(\mathbf{x},t)=v(\mathbf{x},t)=0, & \mathbf{x}\in\partial\Omega, \\ v(\mathbf{x},0)=v_0(\mathbf{x}), \quad u(x,0)=u_0(\mathbf{x}), & \end{cases} \tag{2.1}$$

where $s=S(u,v,\mathbf{x},t)$. To solve Eq. (2.1), the LBGK model is used for the first diffusion equation, while the second equation is solved using an explicit finite difference scheme. Because the LBGK model has first-order accuracy in time and second-order accuracy in space from the following Chapman-Enskog analysis, the first-order Euler scheme

$$u(\mathbf{x},t+\Delta t)=\Delta t v(\mathbf{x},t+\Delta t)+u(\mathbf{x},t)$$