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A Finite Element Method by Patch Reconstruction for the Quad-Curl Problem Using Mixed Formulations

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Abstract. We develop a high order reconstructed discontinuous approximation (RDA) method for solving a mixed formulation of the quad-curl problem in two and three dimensions. This mixed formulation is established by adding an auxiliary variable to control the divergence of the field. The approximation space for the original variable is constructed by patch reconstruction with exactly one degree of freedom per element in each dimension and the auxiliary variable is approximated by the piecewise constant space. We prove the optimal convergence rate under the energy norm and also suboptimal L^2 convergence using a duality approach. Numerical results are provided to verify the theoretical analysis.

AMS subject classifications: 65M60

Key words: Quad-Curl problem, mixed formulation, patch reconstruction.

1 Introduction

The quad-curl problem arises in many multiphysics simulations, especially in inverse electromagnetic scattering for inhomogeneous media, magnetohydrodynamics and also Maxwell transmission eigenvalue problems, see e.g., [1, 3, 4, 6, 9, 16, 19, 23]. Therefore, it is important to design highly efficient and accurate numerical methods for quad-curl problems.

Finite element methods (FEMs) are widely used for solving partial differential equations. The presence of the quad-curl operator makes it difficult and challenging to design conforming finite element spaces for quad-curl problems. We refer to [10,21,22] for some

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recent works on constructing $H(\text{curl}^2)$-conforming finite element spaces in two and three dimensions. Due to the difficulties in discretizing the quad-curl operator, much attention has been paid to nonconforming elements, such as Nédélec 's elements and completely discontinuous piecewise polynomials. We refer to [7, 9, 24] and the references therein for works of this type. Another approach is dedicated to mixed formulations. Some discussions can be found in [18,19,23]. Specifically, [23] reduces the original problem to systems of low-order equations by introducing intermediate variables, which makes the solution easier to approximate.

In this paper, we propose a mixed discontinuous Galerkin finite element method for the quad-curl problem with a divergence-free variable. A significant drawback of the DG space is the large number of degrees of freedom, which results in high computational costs. This drawback is a matter of concern. We follow the methodology in [11–14] to apply the patch reconstruction finite element method to the quad-curl problem. The construction of the approximation space includes creating an element patch for each element and solving a local least squares problem to obtain a polynomial basis function locally. Methods based on the reconstructed spaces are called reconstructed discontinuous approximation methods, which can approximate functions to high-order accuracy while inheriting the flexibility on the mesh partition. One advantage of this space is that it has very few degrees of freedom, which gives high approximation efficiency. The reconstructed space is a subspace of the standard DG space, so we can borrow ideas from the interior penalty formulations to solve the quad-curl problem. For the auxiliary variable, we use the piecewise constant space as the approximation space. Therefore, the mixed system do not grow much in size compared to the original system. By adding penalty terms for both spaces, we do not need the two spaces to satisfy the discrete inf-sup condition. We prove the convergence rates under the energy norm and the L^2 norm, and numerical experiments are conducted to verify the theoretical analysis and show that our algorithm is simple to implement and can reach high-order accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce the quad-curl problem with div-free condition and give the basic notations about the Sobolev spaces and the partition. In Section 3, we establish the reconstruction operator and the corresponding approximation space. Some basic properties of the reconstruction are also proven. In Section 4, we describe the mixed finite element method for the quad-curl problem, and prove that the convergence rate is optimal with respect to the energy norm and suboptimal with respect to the L^2 norm. In Section 5, we carry out some numerical examples to validate our theoretical results and show high-order accuracy of our method. Finally, a brief conclusion is given in Section 6.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ be a bounded polygonal (polyhedral) domain with a Lipschitz boundary $\partial\Omega$.