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An Explicit Superconvergent Weak Galerkin Finite Element Method for the Heat Equation

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Abstract. An explicit two-order superconvergent weak Galerkin finite element method is designed and analyzed for the heat equation on triangular and tetrahedral grids. For two-order superconvergent P_k weak Galerkin finite elements, the auxiliary inter-element functions must be P_{k+1} polynomials. In order to achieve the superconvergence, the usual H^1-stabilizer must be also eliminated. For time-explicit weak Galerkin method, a time-stabilizer is added, on which the time-derivative of the auxiliary variables can be defined. But for the two-order superconvergent weak Galerkin finite elements, the time-stabilizer must be very weak, an H^{-2}-like inner-product instead of an L^2-like inner-product. We show the two-order superconvergence for both semi-discrete and fully-discrete schemes. Numerical examples are provided.

AMS subject classifications: 65N15, 65N30

Key words: Parabolic equations, finite element, weak Galerkin method, polytopal mesh.

1 Introduction

The weak Galerkin (WG) finite element method [32] solves numerically partial differential equations with discontinuous polynomials on polytopal meshes. The method has been developed for many partial differential equations, cf. [2–4, 9, 11–16, 18–21, 24, 25, 27–30, 32, 34, 35, 39, 41, 43, 44, 47, 50–52].

In the P_k weak Galerkin (WG) finite element method, auxiliary variables of P_{k-1}, or P_k, or P_{k+1} polynomials are introduced on the inter-element edges or face-polygons. The continuity of the these discontinuous polynomials are enforced weakly by an H^1-stabilizer. However, for the two-order superconvergent WG finite element method in this manuscript, the auxiliary variables must be of P_{k+1} polynomials on triangular or tetrahedral meshes. Also the H^1-stabilizer must be eliminated from the weak formulation. The

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weak continuity on the discontinuous polynomials is achieved by the weak gradient of properly chosen high-degree polynomials, cf. [36–38, 40, 42, 45, 46].

For time-dependent problems, for a long time, only the implicit WG finite element methods can be applied, cf. [1, 5–8, 10, 17, 22, 26, 48, 49, 53–56]. This is because the time direction updates of the auxiliary variables are not possible. Recently, [33] designs an explicit WG finite element method for the parabolic problems where an L^2-stabilizer is added to the weak formulation. The time derivative is applied inside this L^2-stabilizer so that time-direction updates of the auxiliary variables are consistent.

In this work, we extend the explicit WG finite element method of [33] to the two-order superconvergent WG finite element method, on triangular or tetrahedral meshes. In order to obtain the two-order superconvergence, the L^2-like time-stabilizer in [33] is replaced by a very weak $H^{-2}(\Omega)$-like stabilizer. Of course, the underlying WG finite element space must be a two-order superconvergent one, i.e., in this paper, the P_k-P_{k+1}-P_{k+1}^d WG finite element on triangular or tetrahedral meshes, where the first P_k stands for the polynomial space inside each element, the second P_{k+1} stands for the polynomial space on each edge/triangle between two elements, and the third P_{k+1}^d stands for the discrete space of the weak gradient on each element. Both semi-discrete and fully-discrete numerical schemes are analyzed. Two-order above the optimal-order error estimates are established for both schemes in H^1-norm. Numerical experiments are presented, confirming the theory.

2 Semi-discrete scheme

2.1 The WG finite element space

Let $\mathcal{T}_h$ be a quasi-uniform triangular or tetrahedral mesh of the domain Ω. Denote by $\mathcal{E}_h$ the set of all edges or triangles in $\mathcal{T}_h$ and $\mathcal{E}_h^0=\mathcal{E}_h\backslash\partial\Omega$ the set of all interior edges or triangles. For each element $T\in\mathcal{T}_h$, denote by h_T its diameter and $h=\max_{T\in\mathcal{T}_h}h_T$ the mesh size of $\mathcal{T}_h$. Denote by $P_k(T)$ the space of polynomials of degree less than or equal to k on T. We denote the piecewise bilinear forms in short notations,

$$
\begin{aligned}
(u_0,v_0)_{\mathcal{T}_h}&=\sum_{T\in\mathcal{T}_h}(u_0,v_0),\\
\langle u_b,v_b\rangle_{\partial\mathcal{T}_h}&=\sum_{T\in\mathcal{T}_h}\sum_{e\in T}\langle u_b,v_b\rangle_e.
\end{aligned}
$$

The P_k WG finite element space V_h on the mesh $\mathcal{T}_h$ is defined as follows, $k\geq 1$,

$$
V_h=\{v=\{v_0,v_b\}:v_0|_T\in P_k(T),v_b|_e\in P_{k+1}(e),\ e\subset\partial T,\ \forall T\in\mathcal{T}_h\}. \tag{2.1}
$$

We note that a function $v\in V_h$ has a single value auxiliary v_b on the two sides of $e\in\mathcal{E}_h$. The subspace of V_h consisting of functions with vanishing v_b on $e\subset\partial\Omega$ is denoted as V_h^0.