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High Order Well-Balanced Finite Difference AWENO Scheme for Ripa and Pollutant Transport Systems

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Abstract. The Ripa model consists of the shallow water equations and terms which take the horizontal temperature fluctuations into account. The pollutant transport model describes the transport and diffusion of pollutants in shallow water flows. Both models admit hydrostatic solutions in which the gradient of the flux term is exactly balanced by the source term on the right-hand side. In this paper, we write both models in a unified form and propose a well-balanced fifth-order finite difference alternative weighted essentially non-oscillatory (AWENO) scheme, which allows using arbitrary monotone, Lipschitz continuous and consistent numerical flux. For illustration purposes, the Lax-Friedrichs flux is employed. In order to design a well-balanced AWENO scheme, reformulation of the source term and linearization of the WENO interpolation operator are made. The well-balancedness of the proposed method will be analysed theoretically in this paper. Numerical examples verify the well-balanced property, high-order accuracy and effectiveness of our approach.

AMS subject classifications: 35L65, 65M06, 76M20

Key words: AWENO scheme, well-balanced, Ripa model, pollutant transport model.

1 Introduction

The shallow water equations (SWEs) are attractive among researchers due to their wide range of applications in describing the flood waves, dam breaks and tidal flows in coastal regions. A tremendous amount of numerical approaches for the shallow water equations

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were developed in the recent literature, and readers are referred to [1, 9, 11–14, 26, 28, 36] and references therein.

In this paper, we study two models related to the shallow water equations, i.e., the Ripa model and the pollutant transport model. The Ripa model, which was introduced in [17, 31, 32] to model ocean currents when temperature fluctuations play an important role, consists of the SWEs and terms which take the horizontal temperature fluctuations into account. The governing equations of the systems are derived by integrating vertically the velocity fields, density, and horizontal pressure gradient in each layer of the multi-layer model. Since the motion and behavior of the currents are influenced by forces such as temperature acting on the water, a horizontal temperature gradient is introduced to represent the variation in fluid density. This demonstrates the importance of the study on the Ripa model for understanding real-world phenomena. The pollutant transport model, which was studied in [2, 10], comprises SWEs and terms which concern transport of a passive pollutant in the flow modeled by the Saint-Venant system. Due to the fact that the transport of pollutants in rivers, reservoirs, oceans and some coastal areas may have harmful effects on the environment, it is necessary to predict the transport of pollutants, find their location and concentration, and provide accurate and reliable estimates, which would significantly help to improve the environment.

For simplicity, the Ripa model and the pollutant transport model can be written into a unified form,

$$\frac{\partial \mathbf{Q}}{\partial t}+\nabla\cdot\mathbf{F}(\mathbf{Q})=\mathbf{S}(\mathbf{Q},b), \tag{1.1}$$

where $\mathbf{Q}$, $\mathbf{F}(\mathbf{Q})$ and $\mathbf{S}(\mathbf{Q},b)$ are vectors of conservative variables, flux and source, respectively, with

$$\mathbf{Q}=\begin{pmatrix} h \\ h\mathbf{u} \\ h\xi \end{pmatrix},\quad \mathbf{F}(\mathbf{Q})=\begin{pmatrix} h\mathbf{u} \\ h\mathbf{u}\otimes\mathbf{u}+\frac{1}{2}gh^2\phi(\xi)\mathbf{I}_d \\ h\xi\mathbf{u} \end{pmatrix},\quad \mathbf{S}(\mathbf{Q},b)=\begin{pmatrix} 0 \\ -gh\phi(\xi)\nabla b \\ 0 \end{pmatrix}. \tag{1.2}$$

with

$$\xi=\begin{cases} \theta, & \text{Ripa model}, \\ T, & \text{Pollutant transport model}, \end{cases}\qquad \phi(\xi)=\begin{cases} \theta, & \text{Ripa model}, \\ 1, & \text{Pollutant transport model}, \end{cases}$$

where d is the dimension of the space, h denotes the water height, $\mathbf{u}$ is the water velocity, θ is the temperature, T is the pollutant concentration, g is the gravitational constant and $b=b(x)$ denotes the bottom topography. Here we consider the dimensionless model. In addition, $h+b$ stands for the water surface. A notable feature of the system (1.1) is that both models admit the following steady state solutions,

$$h+b=\text{constant},\quad \mathbf{u}=0,\quad \xi=\text{constant}. \tag{1.3}$$

This is usually called *C-property*, where "C" stands for "constant". At the discrete level, a scheme which preserves the C-property is referred to as a *well-balanced* scheme. The