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A Genuinely 2D Well-Balanced Method for Elastic-Plastic Flow in Cylindrically Symmetric Coordinates

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Abstract. In this work, a genuinely two-dimensional method is proposed for elastic-plastic flow in cylindrically symmetric coordinates. The numerical fluxes are obtained by constructing a genuinely two-dimensional approximate Riemann solver that incorporates consideration of the geometric source term and elastic-plastic transition. The resultant genuine 2D numerical flux combines one-dimensional numerical flux in the central region of the cell edge and two-dimensional flux in the cell vertex region to take wave interaction into account. To deal with the geometry singularity, we establish the equations with removed singularity. For a given stationary state, we prove that the present method is well-balanced. Several numerical tests are presented to verify the performances of the proposed method. The numerical results demonstrate the credibility of the present method.

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Key words: Elastic-plastic flow, cylindrically symmetric coordinates, well-balanced, two-dimensional approximate Riemann solver.

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1 Introduction

Numerous numerical simulations of weapon physics involve elastic-plastic flow [1–19]. One particular area of interest is the study of Armor Piercing (AP) projectiles. Many of those problems are cased into the cylindrically coordinate for less expensive computation. The majority of literature on numerical simulation of elastic-plastic flow involving cylindrically coordinates is mainly based on Lagrangian or ALE methods [20–23]. The advantage of those methods is natural high resolution and inherent flow adaptiveness to

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the material interface. However, they struggle with handling significant mesh and interface deformations. In contrast, the Eulerian coordinate system [24] employs a fixed computational grid and avoids the aforementioned issues. Based on this, this paper aims to investigate the elastic-plastic flow in cylindrically coordinates using the Eulerian framework.

In the cylindrically coordinate system, the elastic-plastic flow equations are accompanied by a geometric source term. The geometric source term introduces two challenges. The first one pertains to the singularity at the symmetry axis. When a shock wave reaches the symmetry axis and reflects outward, we cannot fully explain how the geometric source term affects the solution due to singularity [25,26]. The second challenge involves considering the influence of the geometric source term in the design of the numerical method. Although one-dimensional Riemann solvers have been the cornerstone of constructing multi-dimensional solid solvers via space operator splitting [7,10,11,16,27,28], some researchers have always believed that one-dimensional exact/approximate Riemann solvers lose much of physical characteristics in multi-dimensional problems [29–35]. Thus, a genuinely two-dimensional Riemann problem should be considered for simulating two-dimensional elastic-plastic flows [36]. The inclusion of the geometric source term adds complexity to the construction of a genuinely two-dimensional solver.

In this paper, to deal with the geometry singularity, we assume that the physical solution at $r=0$ is well defined. By applying Lopita's law, the governing equations can be transformed into a singularity-free system of equations. Subsequently, drawing upon previous research [37, 38], a reflection boundary condition is developed at the symmetry axis to ensure that the velocity in the r-direction remains naturally 0 at $r=0$. Furthermore, by integrating the governing equations over a three-dimensional control volume [21,39,40], we propose a framework for constructing numerical scheme in the cylindrically coordinate system. This framework ensures the strict preservation of mass, momentum, and energy conservation. To enhance the stability of the scheme, a semi-implicit time discretization is employed for the source term [37]. Importantly, we demonstrate that the numerical method based on this framework is well-balanced for a given stationary state.

To implement the framework, a general strategy is presented in constructing a genuinely two-dimensional approximate Riemann solver for the elastic-plastic flow, which incorporates the geometric source term and considers the elastic-plastic transition. For the construction of the genuine 2D numerical flux, a corresponding 2D elastic-plastic approximate solver is developed with considering the geometric source term. The resulting genuine two-dimensional flux comprises a 1D numerical flux and two 2D numerical fluxes for taking into consideration of the wave interaction at the cell vertex region. To address the elastic-plastic transition, the solid material is initially assumed to be in an elastic state, and the corresponding 1D and 2D numerical fluxes are devised. If the von Mises condition is violated, the initial state is raised to the elastic limit, and the associated 1D and 2D numerical fluxes are recalculated to accommodate the transition. The stress is updated using the velocity obtained from the approximate Riemann solver.