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Convergence Analysis of a BDF Finite Element Method for the Resistive Magnetohydrodynamic Equations

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Abstract. In this paper we propose and analyze a numerical scheme coupling a second-order backward differential formulation (BDF) and the finite element method (FEM) to solve the incompressible resistive magnetohydrodynamic (MHD) equations. In the discrete scheme, the pressure variable in the fluid field equation is computed through a Poisson equation, and a linear and decoupled method is adopted to separate both the magnetic and the fluid field functions from the original system. As a result, the original system is divided into several sub-systems for which the numerical solutions can be obtained efficiently. We prove the unique solvability, the unconditional energy stability, and particularly optimal error estimates for the proposed scheme. Numerical results are presented to validate the theory of the scheme.

AMS subject classifications: 65M60, 65M12

Key words: Resistive MHD equations, finite element methods, BDF decoupled scheme, unconditional energy stability, optimal error estimates.

1 Introduction

The MHD system describes the interaction between the conductive fluids and the electromagnetic fields [16]. It has been widely applied to the industry production, such as liquid-metal processing, and its numerical solutions are of great significance in science and engineering [45]. This model is governed by the Navier–Stokes equations and the Maxwell equations through the Ohm's law and the Lorentz force. Physically, in order to consider the further effect of magnetic fields, one can introduce a fourth-order curl operator on the magnetic fields into the standard incompressible MHD equations, arriving at

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the following so-called incompressible resistive MHD system [65]

$$\partial_t \boldsymbol{H} - \nabla\times(\boldsymbol{u}\times\boldsymbol{H}) + \frac{\eta}{\mu_0}\nabla\times(\nabla\times\boldsymbol{H}) + \frac{\eta_2}{\mu_0}\nabla\times(\nabla\times(\nabla\times(\nabla\times\boldsymbol{H}))) = \boldsymbol{0}, \tag{1.1a}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p + \frac{1}{\mu_0}\boldsymbol{H}\times(\nabla\times\boldsymbol{H}) = \boldsymbol{0}, \tag{1.1b}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1.1c}$$

over $\Omega\times(0,T]$, where Ω is a bounded and convex polygonal domain in $\mathbb{R}^2$ (polyhedral domain in $\mathbb{R}^3$), and T is a constant representing the final time. Here, the unknowns $\boldsymbol{u}$, $\boldsymbol{H}$ and p denote the velocity field, the magnetic filed, and the pressure variable, respectively. The constant η represents the resistivity, η_2 is the hyper-resistivity, μ is the viscosity of the fluid and μ_0 stands for the magnetic permeability of free space. The initial and boundary conditions are given by

$$\boldsymbol{H}|_{t=0} = \boldsymbol{H}_0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{1.2a}$$

$$\boldsymbol{H}\times\boldsymbol{n} = \boldsymbol{0}, \quad (\nabla\times(\nabla\times\boldsymbol{H}))\times\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega\times(0,T]. \tag{1.2b}$$

It is assumed that the initial data satisfies

$$\nabla\cdot\boldsymbol{H}_0 = \nabla\cdot\boldsymbol{u}_0 = 0. \tag{1.3}$$

By taking the divergence of (1.1a), we have $\partial_t\nabla\cdot\boldsymbol{H} = 0$, which together with the above divergence-free initial condition indicates that $\nabla\cdot\boldsymbol{H} = 0$ for any $t > 0$.

Apparently, taking hyper-resistivity coefficient $\eta_2 = 0$ would reduce the original system (1.1a)-(1.1c) into the standard incompressible MHD system. There have been already many works dedicated to regularity analysis of the incompressible MHD system [23, 36, 37, 48]. Concerning finite element methods for the MHD system, many research efforts have been devoted to the use of the $H^1(\Omega)$ conforming elements, since the weak solutions of the system are located in $H^1(\Omega)$. In [22], Gunzburger et al. proposed a numerical scheme and analyzed optimal error estimates for the stationary MHD system by $H^1(\Omega)$ conforming elements. The similar results were obtained for the time-dependent MHD model in [24]. Li et al. developed a strongly convergent finite element scheme based on the $H^1(\Omega)$ conforming elements in general domains, which may be nonconvex, nonsmooth and multi-connected, without any mesh restriction [30]. Wang et al. designed a second-order temporally accurate finite element scheme with the $H^1(\Omega)$ conforming elements, and provided a rigorous proof on optimal error estimates [47]. More works about $H^1(\Omega)$ conforming elements are referred to [25,47,52,58,60] and references therein. An apparent difference between the standard MHD system and the resistive MHD system is the appearance of the fourth-order curl operator, for which many numerical schemes have been proposed and analyzed. Zheng et al. utilized a nonconforming finite element involving a small number of degrees of freedom for its solution [65]. Sun proposed a mixed finite element method by introducing an intermediate