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Spectral Scheme for Nonlinear Volterra Integro-Differential Equation

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Abstract. Nonlinear problems widely exist in many aspects of the natural field. The nonlinear situation makes it difficult for most existing solvers to deal with. Therefore, constructing an efficient and accurate solver is a challenge. In this paper, a Legendre spectral method is developed for the nonlinear Volterra integro-differential equation. The error analysis is also provided to justify the spectral rate of convergence for the er-rors of approximate solution and approximate derivative decay exponentially in both the L^2 norm and the infinity norm. In the end, numerical results are displayed to con-firm the effectiveness of the Legendre spectral analysis.

AMS subject classifications: 65R20, 45E05

Key words: Legendre spectral method, nonlinear Volterra integro-differential equation, numerical simulation.

1 Introduction

Nonlinear problems widely exist in many aspects of the natural field, such as heat transfer problem [11], Hamilton equation in physics [16], nonlinear reaction-diffusion equation in hydrodynamics [4,9], Cauchy problem [1] and nonlinear Volterra equations in biology [17,21], etc. The obvious characteristic of nonlinearity is non superposition, which makes the research of nonlinear problems more complex. It is this complexity that inspires many mathematicians to study deeply. However, most nonlinear problems are often difficult to obtain accurate solutions, and approximate solutions are required. Therefore it is important to construct high-efficiency and high-precision numerical calculation

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methods, and as far as we know, spectral method is a global numerical method which provides exponential rate of error estimate. The super-convergence of spectral method has attracted an amount of attention, especially for Volterra equations. It is worth to mentioning that a breakthrough is made in [21], in which the authors constructed a spectral approximation to nonlinear Volterra integral equations. But this is a remarkable contrast to the number of work on the linear Volterra equations. Here we just list some representative work [6,7,13,26,27]. On the other hand, nonlinear situation seems to be more useful tool to model the behavior of the real world and studies of numerical methods for nonlinear Volterra equations still under development [10,14,15,19,20,22–24], but some of them are not the most suitable methods. There is no doubt that research on efficient numerical methods for nonlinear Volterra integro-differential equations has become significant and valuable. Now we give the equation to be studied bellow

$$y'(\tau)=A(\tau,y(\tau))+\int_0^\tau K(\tau,\xi,y(\xi))d\xi+g(\tau), \tag{1.1}$$

with the initial condition as

$$y(0)=y_0, \tag{1.2}$$

where $y(\tau)$ is an unknown function and assumed to be sufficiently smooth, $\tau\in[0,T]$. $A(\tau,y(\tau))$, $K(\tau,\xi,y(\xi))$, $g(\tau)$ called coefficient function, kernel function and source function, respectively, they are given functions and assumed to be sufficiently smooth on their respective domains. $A(\tau,y(\tau))$ and $K(\tau,\xi,y(\xi))$ are Lipschitz continuous with y. Nonlinear Volterra equation is utilized to describe a variety of phenomena in domains like fluid mechanics [8] and population model [17]. Obviously, the existence and the uniqueness of the solution for the two-dimensional weakly singular Volterra integral equations with delays are valid from the theory of ODE.

The rest of the work is arranged as follows. In the subsequent section, we describe the Legendre spectral scheme. Five lemmas for the convergence analysis are introduced in Section 3 and the spectral analysis for nonlinear Volterra integro-differential equation is carried out in Section 4. Section 5 is used to confirm the theoretical analysis by examples. Conclusions are given in the last section.

2 Legendre spectral analysis

For the sake of applying Legendre spectral method, we transform the variables in (1.1) as follows. Let

$$\tau=\frac{T(1+x)}{2},\quad \xi=\frac{T(1+s)}{2},$$

and (1.1)-(1.2) can be written as

$$u'(x)=a(x,u(x))+\int_{-1}^x \hat{K}(x,s,u(s))ds+f(x), \tag{2.1a}$$

$$u(-1)=u_{-1}, \tag{2.1b}$$