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Mass-, and Energy-Preserving Sine Pseudo-Spectral Schemes with High-Accuracy for the Coupled Nonlinear Schrödinger Equation

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Abstract. In the paper, we consider the coupled nonlinear Schrödinger equation with high degree polynomials in the energy functional that cannot be handled by using the newly proposed quadratic auxiliary variable method. Therefore, we develop the multiple quadratic auxiliary variable approach to deal with coupled systems and construct high-accuracy structure-preserving schemes for the equation. To fix the idea, we first apply the multiple quadratic auxiliary variable approach to the equation and obtain an equivalent system that possesses the original energy and mass. Then, a family of high-accuracy structure-preserving schemes that can conserve the mass and energy is derived by applying the Gauss collocation method and sine pseudo-spectral method to approximate the system in time and space. The given schemes have high-accuracy in time and can both inherit the mass and Hamiltonian energy of the system. Ample numerical results are given to confirm the accuracy and conservation of the developed schemes at last.

AMS subject classifications: 65M06, 65M70

Key words: Multiple quadratic auxiliary variable, structure-preserving, Gauss collocation method, Sine pseudo-spectral method, coupled nonlinear Schrödinger equation.

1 Introduction

The coupled nonlinear Schrödinger (CNLS) equation is an important model in the study of optical solitons, it can describe the dynamics of ultrashort optical pulses in an optical

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system including two or more interactions [19]. In this paper, we aim to present and analyze high-order numerical schemes with conservation properties for the CNLS equation that can be written in the form [26]

$$\mathrm{i}\varphi_t+\vartheta\mathcal{L}\varphi+\zeta\big(|\varphi|^2+\varepsilon|\phi|^2\big)\varphi=0, \qquad \boldsymbol{x}\in\Omega, \quad 0\le t\le T, \tag{1.1a}$$

$$\mathrm{i}\phi_t+\vartheta\mathcal{L}\phi+\zeta\big(|\phi|^2+\varepsilon|\varphi|^2\big)\phi=0, \qquad \boldsymbol{x}\in\Omega, \quad 0\le t\le T, \tag{1.1b}$$

with the initial conditions

$$\varphi_0=\varphi(x,0), \quad \phi_0=\phi(x,0),$$

and boundary conditions

$$\varphi(\boldsymbol{x},t)|_{\partial\Omega}=\phi(\boldsymbol{x},t)|_{\partial\Omega}=0,$$

where i is the imaginary unit root, $\mathcal{L}$ is the Laplacian operator, $\boldsymbol{x}\in\Omega\subseteq\mathbb{R}^d$ $(d=1,2)$, and ϑ, ζ, ε are real constants, $\varphi(\boldsymbol{x},t)$ and $\phi(\boldsymbol{x},t)$ are complex valued functions, φ_0 and ϕ_0 are initial functions. The CNLS equation possesses the following physical invariants that do not change with time [28]

$$\mathcal{M}(t):=\int_\Omega \big(|\varphi|^2+|\phi|^2\big)dx\equiv\mathcal{M}(0), \tag{1.2a}$$

$$\mathcal{H}(t):=\vartheta\int_\Omega\Big[(\nabla\varphi)^2+(\nabla\phi)^2\Big]dx-\frac{\zeta}{2}\int_\Omega\Big(|\varphi|^4+|\phi|^4+2\varepsilon|\varphi|^2|\phi|^2\Big)dx\equiv\mathcal{H}(0), \tag{1.2b}$$

where $\mathcal{M}$ is the mass and $\mathcal{H}$ is the Hamiltonian energy of the system.

By setting $\varphi=u+\mathrm{i}v$, $\phi=p+\mathrm{i}q$, the original system (1.1a)-(1.1b) can be rewritten as the following real system

$$\begin{cases} u_t+\vartheta\mathcal{L}v+\zeta\Big[u^2+v^2+\varepsilon(p^2+q^2)\Big]v=0,\\ v_t-\vartheta\mathcal{L}u-\zeta\Big[u^2+v^2+\varepsilon(p^2+q^2)\Big]u=0,\\ p_t+\vartheta\mathcal{L}q+\zeta\Big[p^2+q^2+\varepsilon(u^2+v^2)\Big]q=0,\\ q_t-\vartheta\mathcal{L}p-\zeta\Big[p^2+q^2+\varepsilon(u^2+v^2)\Big]p=0. \end{cases} \tag{1.3}$$

According to the variational derivative formula [30], the CNLS system (1.3) can be further expressed by an infinite-dimensional Hamiltonian system

$$\frac{dy}{dt}=\mathcal{S}\frac{\delta H(y)}{\delta y}, \quad \mathcal{S}=\begin{pmatrix} O & I_2\\ -I_2 & 0\end{pmatrix}, \tag{1.4}$$