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Mathematical Modeling and Hyers-Ulam Stability for a Nonlinear Epidemiological Model with Φ_p Operator and Mittag-Leffler Kernel

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Abstract. This paper investigates a novel nonlinear singular fractional SI model with the Φ_p operator and the Mittag-Leffler kernel. The initial investigation includes the existence, uniqueness, boundedness, and non-negativity of the solution. We then establish Hyers-Ulam stability for the proposed model in Banach space. Optimal control analysis is performed to minimize the spread of infection and maximize the population of susceptible individuals. Finally, the theoretical results are supported by numerical simulations.

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Key words: Epidemiological model, fractional differential equations, p-Laplacian operator, numerical approximations, optimal control.

1 Introduction

Fractional-order models have attracted considerable interest from researchers in a wide variety of disciplines. Over the past two decades, these models have found applications in a wide variety of scientific and engineering fields, including modern physics, signal theory, control theory, hydrodynamics, viscoelastic theory, fluid dynamics, set theory, computer networks, biology, etc. Relevant literature on these topics can be found in the works [15,17,20,26–28,30,35,37,41].

Recently, several researchers have studied *fractional differential equations* (FDEs) with singularities using various mathematical methods. For example, Bai and Qiu [7] established the *existence and uniqueness* (EU) of the solution to a nonlinear singular *boundary value problem* (BVP) of FDEs using the Krasnoselskii and Leray-Schauder fixed point

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theorems. They also demonstrated applications to underscore their results. Agarwal, O'Regan and Staněk [2] studied the EU for a singular fractional BVP using the Riemann-Liouville fractional derivative. Bai and Fang [6] studied a singular nonlinear coupled system of FDEs, using Leray-Schauder and Krasnoselskii fixed point techniques for the EU of the solution. Vong [40] studied FDEs with singularity and non-local boundary conditions using the Schauder fixed-point approach and upper-lower solution techniques. Pu et al. [36] studied positive solutions of a multipoint BVP with singularity and applied their results to a specific example. Khan, Chen and Sun [25] studied nonlinear FDEs with singularity and p-Laplacian to establish the EU of the solution and performed stability analysis.

Mathematical models have long been indispensable tools for understanding and predicting the dynamics of infectious diseases. By quantifying the complex interactions between pathogens and populations, these models enable researchers and policymakers to gain insight into the spread of disease and evaluate potential control strategies. Some of the pioneering work in epidemic modeling can be attributed to Kermack and McKendrick [23], who introduced the SIR model in 1927. This model divided the population into three compartments: susceptible (S), infectious (I), and recovered (R). Using differential equations, the model captured the transitions between these compartments and laid the foundation for subsequent advances in epidemic modeling.

Over time, mathematical models of epidemics have evolved and expanded to incorporate additional complexities. Researchers recognized the importance of accounting for factors such as age structure, spatial heterogeneity, and varying transmission rates. This led to the development of more sophisticated compartmental models, such as the SEIR model, which introduced an exposed compartment [4]. For a recent review on epidemiological models we refer the interested reader to [18] and the references therein.

In addition, spatial epidemic models and network-based models emerged to capture the influence of geographic location and social connectedness on disease spread [22]. In recent years, advanced mathematical techniques have further enhanced the capabilities of epidemic models. Network theory has provided insights into the role of social connections in disease transmission, allowing the exploration of targeted intervention strategies. Nonlinear dynamics and chaos theory have shed light on complex epidemic behavior, including the emergence of periodic outbreaks and bifurcations.

The main goal of this paper is to minimize the number of infected individuals for the *fractional SI model*, which describes the evolution of two compartments: susceptible individuals (S) and infected individuals (I), considering the *Atangana-Baleanu fractional derivative in the Caputo sense* ($\mathcal{ABC}$ fractional derivative, for short) and the Φ_p operator. The inclusion of these operators in our epidemic model offers several compelling motivations. First, the use of the $\mathcal{ABC}$ fractional derivative allows for the inclusion of memory effects in the model and long-range interactions in disease transmission. Traditional derivative operators assume instantaneous changes, which may not accurately capture the dynamics of infectious diseases. By introducing fractional calculus, we can account for the persistence of past infection rates, allowing for a more realistic representation of disease transmission.