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Penalty Finite Element Method for the 2D/3D Unsteady Incompressible Thermomicropolar Fluid Equations

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Abstract. In this paper, a first-order penalty finite element method for the 2D/3D unsteady incompressible thermomicropolar fluid (UITF) equations is considered, which combines the advantage of the penalty method, first-order backward Euler scheme, and implicit or explicit iteration for the nonlinear terms to get a decoupled temporal evolution procedure, which only needs to solve a series of elliptic subproblems. Theoretically, the stability analysis and optimal error estimates of the temporal semi-discrete method are deduced. Furthermore, the classical MINI element pairs are adopted in concrete spatial discrete, and the feasibility of the method is verified by numerical experiments.

AMS subject classifications: 76M10, 65N12, 65N30

Key words: Thermomicropolar fluid equations, penalty method, backward Euler scheme, error estimates.

1 Introduction

The incompressible thermomicropolar fluid equations are first introduced by Eringen in [1], which can be used to describe the flow behavior with additional microscopic rotation effects due to the special structure and/or microscale motions of the fluid materials. Thus, in some sence, the incompressible thermomicropolar fluid equations may be expected to describe more details of flow behavior than the classical thermal Navier-Stokes equations. The incompressible thermomicropolar fluid equations can be viewed as a coupling system of the micropolar fluid equations with heat equations, which exhibits microrotational effects and microrotational inertia under the influence of a thermal field. Therefore, the incompressible thermomicropolar fluid equations can describe more details than the natural convection equations and the micropolar fluid equations. In recent

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years, many scholars have noticed that the equations plays an important role both in the engineering [2] and mathematical literature [3]. However, there is still little research work on the numerical methods of the equations, the research in this paper is urgent and relevant due to the importance in bioengineering. The existence and uniqueness of the solution of the incompressible thermomicropolar fluid equations has been proved in [1,4,5].

Noting that the Galerkin variational problem of the UITF equations is still the saddle-point type [6–8] multi-physics coupling system, which is difficult to be solved numerically, so it's particularly important to find a simple and decoupled method to solve the equations. Since the incompressible thermomicropolar fluid equations can be looked at an extention of the classical Navier-Stokes equations, many difficulties about this equations can be referred to the treatment means of the Navier-Stokes equations. One of the popular methods to overcome the difficulty are to relax the incompressible constraint, such as the penalty method [9–12], projection method [13, 14], pressure stabilization method [15,16] and artificial compressibility method [17,18]. Among them, the penalty method was the simplest and most basic, and it also has the good effect, so the penalty method is mainly considered in this paper.

The penalty method can be worked as a decoupling method, which can easily eliminate the pressure term by using the penalty relation to obtain a penalty system containing only linear velocity term, angular velocity term and temperature term, then directly get the numerical solution of original equations. The method has been adopted by many computational fluid dynamics practitioners to study the utilization [9–11,19]. For example, the fully-discrete penalty method to solve the steady Navier-Stokes equations were given in [9], and the optimal error estimates of the variables were proved. The penalty method based on an Euler implicit or explicit scheme to solve the unsteady Navier-Stokes equations with smooth or non-smooth initial date was proposed in [10], and the optimal error estimates of the numerical velocity and the numerical pressure under the stability condition were obtained. The penalty method to solve the steady micropolar fluid equations were given in [11], a priori properties, existence, uniqueness, stability, and error estimates were proved. And the penalty finite element method to solve the unsteady micropolar fluid equations were obtained in [12], the optimal error estimates for variables of the equations with sufficiently small parameters ε, τ, and h were given.

As far as we know, there is no literature on the penalty method of the UITF equations. The purpose of this paper is to extend the penalty method to the UITF equations and derive the optimal error estimation and temporal discrete of the UITF equations penalty method. The main conclusions of this paper are the optimal error estimation of UITF penalty method in Theorem 4.1 and the optimal error estimation of the penalty method in the temporal semi-discrete scheme in Theorem 5.1.

The paper is organized as follows. In Section 2, some preliminary conclusions of the UITF equations and the penalty method are given. In Section 3 and Section 4, the error behavior of the penalty method for linearized UITF equations and nonlinear UITF equations are analyzed, respectively. In Section 5, the temporal semi-discrete scheme