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Asymptotic Exponential Arbitrage in a Liu-Tang 3-Factor Model of Commodity Futures

Tesfamariam Tadesse Welemical[1], Martin Le Doux Mbele Bidima[2,*] and Jane Akinyi Aduda[3]

[1] *Aksum University, Ethiopia*
[2] *University of Yaoundé I, Cameroon*
[3] *Jomo Kenyatta University of Agriculture and Technology, Kenya*

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Abstract. From a 3-factor model of storable commodities discussed by Liu and Tang (2010), we consider a cash market model such as futures exchange with a single futures contract on one such commodities and a money market account. After verifying that this model is arbitrage-free and incomplete in any finite time horizon or delivery date, we show that there still exists a possibility to generate exponentially growth risk-less profit in long term; a form of asymptotic arbitrage conjectured by Föllmer and Schachermayer (2008) and first solved by Mbele Bidima and Rásonyi (2012) in financial security models. And we find that works in this paper generalize our recent works in Tadesse Welemical *et al.* (2019) on Schwartz's one-factor model of commodity futures.

AMS subject classifications: 91B70, 91G10, 91G15

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1 Introduction

A key challenge for market makers and policymakers is to avoid losses while guaranteeing some (risk-less) profits in long-term economic trading. Such profits were first studied under a concept named "asymptotic arbitrage" by Kabanov and Kramkov [13] in their pioneering work. The concept gained some incredible development over the past decade after the contributing and inspiring works of Föllmer and Schachermayer [8]. It is an emerging theory in modern Mathematical Finance where authors are analyzing existence of arbitrage opportunities (risk-less profits) in long-term trad-

*Corresponding author. *Email:* mmbele@facsciences-uy1.cm (M. L. D. Mbele Bidima)

ing, i.e. when the trading time horizon T tends to infinity, in general (or typical) financial models. This concept is discussed depending on whether classical arbitrage in any finite time horizon appear or not in those models as guaranteed by the First Fundamental Theorem of Asset Pricing (stated in [4, 12]). However, the literature in this subject is still narrow and shows that the analysis of asymptotic arbitrage has been carried out essentially by a limited number of authors in models of financial securities only, see for instance [3, 6, 11, 14, 16, 17].

A couple of years ago [20], we were the first to conduct analysis of asymptotic arbitrage in models of storable commodities, especially in the Schwartz's one-factor model of storable commodity futures. In that article, we carefully started with highlighting the difference between standard securities markets and (storable) commodities market. Indeed, unlike financial securities (such as stocks, bonds, etc.), storable commodities (like oil, gold, coffee, etc.) are characterized by an unavoidable cost of carrying a physical good, also known as the convenience yield. And they are traded in two interrelated markets: the storage (inventory) market where physical commodities are assumed traded at prices called spot prices, and the cash market (typically a futures exchange) where futures contracts on those commodities with maturity time (delivery date) T are traded at time $t \leq T$ with prices known as futures prices.

In Schwartz's one factor model of storable commodities [18], the convenience yield is assumed constant, and this was crucial in our analysis of asymptotic arbitrage in [20]. But in a number of other classical models of (storable) commodities such as [10], the convenience yield is not constant and is modeled as an Ornstein-Uhlenbeck (OU) process, which can take negative values. But Liu and Tang [15, Lemma 1, Theorem 1] proved that under short-selling prohibiting, the negativity of the convenience yield is equivalent to existence of arbitrage opportunities in the storage market in any finite trading time horizon. To overcome such a limitation, the authors of [15] modeled the convenience yield using a Cox-Ingersoll-Ross (CIR) process. As a result, they developed a so-called semi-affine 3-factor model (with factors being the spot price of a commodity, the convenience yield and the short interest rate) under an equivalent martingale measure (EMM) $\mathbb{Q}$ existing in the cash market. Since the CIR process assumes the non-negativity constraint for the underlying (the convenience yield in this case), then this guarantees the requirement of no-arbitrage opportunity in their such storage market model in any finite time horizon.

In this paper, we consider in the section below the so-called Liu-Tang 3-factor commodity futures model (Definition 2.1) which we built on the setup of the Liu-Tang 3-factor model for a storable commodity that follows. We verify in Section 3 that this constructed 3-factor futures model is arbitrage-free and incomplete in any finite time horizon (future delivery date) $T > 0$. Next in Section 4, which is the main part of our article, after recalling the concept asymptotic exponential arbitrage with geometrically decaying failure probability which discussed in [20] for a Schwartz's one factor commodity futures model, we prove in Theorem 4.1 existence of such trading opportunities in the Liu-Tang 3-factor commodity futures model. And we end the paper with a conclusion and some perspectives in Section 5.