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Renormalized Solution of the Relativistic Boltzmann Equation in a Robertson-Walker Spacetime

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Abstract. In this paper, the proof of the global existence of a renormalized or equivalently mild solution of the relativistic Boltzmann equation in a Robertson-Waker space-time is given for an initial value problem with initial data only satisfying the conditions of finite mass, energy, inertia and entropy.

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1 Introduction

We are concerned with global existence of solution of the initial value problem for the following dimensionless relativistic Boltzmann equation (RBE) in a Robertson-Walker space-time [6]

$$\partial_t f + \hat{p} \cdot \nabla_x f - 2\frac{\dot{G}}{G} p \cdot \nabla_p f = Q(f;f), \tag{1.1}$$

where different parts will be addressed below.

About the relativistic case, several authors have studied this problem by taking the Minkowski spacetime as background. Most of results available concern the study of mild solutions [1,3,8,9].

In their seminar paper DiPerna and Lions [1], based on new tools and techniques, have studied the non-relativistic Boltzmann equation. The key concept of their results is the notion of renormalized solution of the transport equation. By the velocity averaging, they permit to show the proof of global existence of weak solution via

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compactness arguments. After this result, the desire to extend this method to the relativistic case becomes a problem. The response came firstly by Dudyński and Ekiel-Jeżewska in [2], in Minkowski spacetime, with an existence proof based on the causality property of the relativistic Boltzmann equation. By modifying the assumptions made on the scattering cross section in [2], more complicated in the relativistic case, Zhenglu [8,9] has given the proof of global existence of renormalized solution for the initial value problem for the relativistic Boltzmann equation using the DiPerna and Lions method's in Minkowski spacetime.

The objective of this paper is to use the same approach as in [1,9] and prove that there exists a global renormalized equivalently mild solution to the large data Cauchy problem for the relativistic Boltzmann in Robertson-Walker spacetime under the condition of initial data f_0 satisfying (2.32), that is

Theorem 1.1. *Let $K(g,\theta)$ be the relativistic collision kernel of the RBE* (2.14)*, and B_r a ball with center at the origin and radius r, $B(g)=\int_{S^2}K(g,\theta)d\Omega$. Assume that*

$$K(g,\theta)\geq 0,\quad a.e.\ in\ \ [0,+\infty)\times S^2,\quad K(g,\theta)\in L^1_{loc}(R^3\times S^2), \tag{1.2}$$

$$\frac{1}{(v^0)^2}\int_{B_r}\frac{B(g)}{v_1^0}dv_1\ \ \rightarrow\ \ 0\quad as\quad |v|\ \ \rightarrow\ \ +\infty,\quad \forall\, r,t\in(0,+\infty). \tag{1.3}$$

Then the RBE (1.1) *has a renormalized or equivalently a mild solution f through initial data f_0 with* (2.32) *satisfying the following properties:*

$$f\in C\big([0,+\infty);L^1(\mathbb{R}^3\times\mathbb{R}^3)\big) \tag{1.4}$$

$$L(f)\in L^\infty\big([0,+\infty);L^1\big(\mathbb{R}^3\times B_r\big)\big),\qquad \forall\, r\in(0,+\infty), \tag{1.5}$$

$$\frac{Q^+(f,f)}{1+f}\in L^\infty\big([0,+\infty);L^1\big(\mathbb{R}^3\times B_r\big)\big),\quad \forall\, r\in(0,+\infty), \tag{1.6}$$

$$\sup_{t\geq 0}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}f(1+\ln f)dx\,dp<+\infty. \tag{1.7}$$

The challenge is the form of the Boltzmann equation in this spacetime. In order to use the DiPerna and Lions method, we base our approach in the transformation of Eq. (1.1) into a different equivalent form using covariant variables as in [5,6]. Then we follow the steps of [1,9]. In this work we use the similar assumptions on the scattering kernel already used in [9], namely

$$K(g,\theta)\geq 0\quad \text{a.e.}, \tag{1.8}$$

$$z(1+z^2)K(z,\theta)\in L^1_{Loc}\big((0,+\infty)\times S^2\big), \tag{1.9}$$

$$\frac{1}{(v^0)^2}\int\int_{B_r\times S^2}\frac{gs^{\frac{1}{2}}K(g,\theta)}{v_1^0}d\Omega dv_1\ \ \rightarrow\ \ 0,\quad |v|\ \ \rightarrow\ \ +\infty,\quad \forall\, r,t\in(0,+\infty), \tag{1.10}$$

where for $r>0$, B_r is a ball with its center in the origin and radius r, and where the other quantities will be specified in the sequel.