Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0079

Vol. **37**, No. 1, pp. 1-29
January 2025

Approximating and Preconditioning the Stiffness Matrix in the GoFD Approximation of the Fractional Laplacian

Weizhang Huang[1] and Jinye Shen[2,*]

[1] *Department of Mathematics, University of Kansas, Lawrence, Kansas, U.S.A.*
[2] *School of Mathematics, Southwestern University of Finance and Economics, Chengdu, P.R. China.*

Received 4 April 2024; Accepted (in revised version) 11 July 2024

Abstract. In the finite difference approximation of the fractional Laplacian the stiffness matrix is typically dense and needs to be approximated numerically. The effect of the accuracy in approximating the stiffness matrix on the accuracy in the whole computation is analyzed and shown to be significant. Four such approximations are discussed. While they are shown to work well with the recently developed grid-over finite difference method (GoFD) for the numerical solution of boundary value problems of the fractional Laplacian, they differ in accuracy, economics to compute, performance of preconditioning, and asymptotic decay away from the diagonal line. In addition, two preconditioners based on sparse and circulant matrices are discussed for the iterative solution of linear systems associated with the stiffness matrix. Numerical results in two and three dimensions are presented.

AMS subject classifications: 65N06, 35R11

Key words: Fractional Laplacian, finite difference approximation, stiffness matrix, preconditioning, overlay grid.

---

1 Introduction

We are concerned with the finite difference (FD) solution of the boundary value problem (BVP) of the fractional Laplacian,

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega, \\ u = 0, & \text{in } \Omega^c, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `jyshen@swufe.edu.cn` (J. Shen), `whuang@ku.edu` (W. Huang)

where $(-\Delta)^s$ is the fractional Laplacian with the fractional order $s\in(0,1)$, Ω is a bounded domain in $\mathbb{R}^d$ $(d\geqslant 1)$, $\Omega^c\equiv\mathbb{R}^d\backslash\Omega$ is the complement of Ω, and f is a given function. The fractional Laplacian can be expressed in the singular integral form as

$$(-\Delta)^s u(\vec{x})=\frac{2^{2s}s\Gamma(s+\frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(1-s)}\text{ p.v. }\int_{\mathbb{R}^d}\frac{u(\vec{x})-u(\vec{y})}{|\vec{x}-\vec{y}|^{d+2s}}d\vec{y}, \tag{1.2}$$

or in terms of the Fourier transform as

$$(-\Delta)^s u=\mathcal{F}^{-1}(|\vec{\xi}|^{2s}\mathcal{F}(u)), \tag{1.3}$$

where p.v. stands for the Cauchy principal value, $\Gamma(\cdot)$ is the gamma function, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively. When Ω is a simple domain such as a rectangle or a cube, BVP (1.1) can be solved using finite differences on a uniform grid (see, e.g., [26,28]). When Ω is an arbitrary bounded domain (including a simple domain), BVP (1.1) can be solved using the recently developed grid-overlay finite difference (GoFD) method with a simplicial mesh [28] or a point cloud [40]. Generally speaking, the stiffness matrix in FD approximations is dense and needs to be approximated numerically. The effect of the accuracy in the approximation on the accuracy in the numerical solution of BVP (1.1) has not been studied in the past. A main objective of the present work is to study this important issue for the numerical approximation of the fractional Laplacian. It will be shown that the effect is actually significant. As a result, it is necessary to develop accurate and reasonably economic approximations for the stiffness matrix. We will study four approximations. The first two are based on the fast Fourier transform (FFT) with uniform and non-uniform sampling points. The third one is the spectral approximation of Zhou and Zhang [47]. We will discuss a new and fast implementation of this approximation and derive the asymptotic decay rate of its entries away from the diagonal line. The last one is a modification of the spectral approximation. Properties of these approximations are summarized in Table 3. In addition, we will study two preconditioners based on sparse and circulant matrices for the iterative solution of linear systems associated with the stiffness matrix.

For the purpose of numerical verification and demonstration, we consider an example of BVP (1.1) with the analytical exact solution (cf. [21, Theorem 3]),

$$\Omega=B_1(0),\quad f=1,\quad u=\frac{\Gamma(\frac{d}{2})}{2^{2s}\Gamma(1+s)\Gamma(\frac{d}{2}+s)}(1-|\vec{x}|^2)_+^s, \tag{1.4}$$

where $B_1(0)$ is the unit ball centered at the origin. Numerical results will be given in two and three dimensions.

The fractional Laplacian is a fundamental non-local operator in the modeling of anomalous dynamics; see, for example, [6,31,33] and references therein. A number of numerical methods have been developed, including FD methods [16,18,19,26,30,31,34,37,43,45,46], finite element methods [1–4, 10, 11, 22, 44], spectral methods [32, 41, 47], discontinuous