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Newton's Method and Its Hybrid with Machine Learning for Navier-Stokes Darcy Models Discretized by Mixed Element Methods

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Abstract. This paper focuses on discussing Newton's method and its hybrid with machine learning for the steady state Navier-Stokes Darcy model discretized by mixed element methods. First, a Newton iterative method is introduced for solving the relative discretized problem. It is proved technically that this method converges quadratically with the convergence rate independent of the mixed element mesh size, under certain standard conditions. Later on, a deep learning algorithm is proposed for solving this nonlinear coupled problem. Following the ideas of an earlier work by Huang, Wang and Yang (2020), an Int-Deep algorithm is constructed by combining the previous two methods so as to further improve the computational efficiency and robustness. A series of numerical examples are reported to show the numerical performance of the proposed methods.

AMS subject classifications: 65N12, 65N15, 65N22, 65N30

Key words: Navier-Stokes Darcy model, deep learning, Newton iterative method, Int-Deep method, convergence analysis.

1 Introduction

The Navier-Stokes Darcy model is frequently encountered in various industrial engineering scenarios, for instance, in groundwater [10,14,27], flow in porous media [2,3], industrial filtrations [18] and so on. Due to its importance, many numerical methods have been developed and analyzed for the nonlinear coupled model, which can be roughly divided into two categories. The first one focuses on numerically solving the coupled

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system directly. Historically, Girault and Riviére [17] propose and analyze a discontinuous Galerkin method for the Navier-Stokes Darcy model. Following [17], Chidyagwai and Riviére [12] develop a hybrid coupled method, where the continuous finite elements are employed in the free flow region and the discontinuous Galerkin finite elements in the porous medium region, respectively. Wu and Mei [32] discretize the model with a non-conforming finite volume element method. Cesmelioglu and Rhebergen present and analyze in [11] a strongly conservative hybridizable discontinuous Galerkin scheme. The other strategy is to decouple the model first and then numerically solve two subproblems separately. In [9], Cai, Mu and Xu propose a decoupled and linearized two-grid algorithm. The two-grid decoupled method is further studied in [25, 36]. In addition, Cai, Huang and Mu also develop a multi-grid algorithm in [8]. He et al. [20] design a domain decomposition method for the Navier-Stokes Darcy model with the BJ interface condition. In order to further improve computational efficiency, Du et al. develop in [16, 31] a series of parallel algorithms based on decoupled model. We mention that all the previous methods are used to solve steady-state Navier-Stokes Darcy models. In fact, there are deep and through works on numerical solution for unsteady state models, and we skip the details due to the limit of space.

Both the direct and decoupling methods require solving nonlinear system of equations globally or locally through iterative methods, e.g. Newton iterative method. However, there are few works on investigating the global convergence analysis of these iterative methods. As far as we know, only Badea, Discacciati, and Quarteroni [4] study Newton iterative method for the Navier-Stokes Darcy model in the case of infinite dimensions. The analysis is rather involved; the key techniques are first reformulating the coupled nonlinear problem as an interface equation and then developing the underlying convergence analysis of Newton iterative method by means of the Kantorovich theorem. However, when applying the previous arguments to the nonlinear system arising from discretization of the Navier-Stokes Darcy model by mixed element methods, it is very hard to show the convergence rate is uniformly bounded with respect to the finite element mesh size h. This study is important for real applications. In fact, if the convergence rate goes to 1 when h goes to zero, then we even cannot observe convergence of the underlying iterative method due to the rounding-off error. The other point to be emphasized is that Newton's method is locally convergent, so it is very challenging to develop a strategy on the choice of initial guesses to ensure convergence. For our problem under discussion, a common way is to set the initial guess as the numerical solution of the corresponding linear Stokes Darcy problem. However, such a choice is not robust with respect to the model coefficients; we refer to numerical results in Subsection 5.1.2 for details. Therefore, it's necessary to develop an effective approach to choosing initial guesses in order to improve the computational performance and robustness of Newton iterative method.

With the above discussion in mind, our study in this paper is twofold. First of all, we analyze in detail the convergence of Newton iterative method for the Navier-Stokes Darcy model discretized by mixed finite element methods. Different from the approach