

An Arbitrary Order Mixed Finite Element Method with Boundary Value Correction for the Darcy Flow on Curved Domains

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Abstract. We propose a boundary value correction method for the Brezzi-Douglas-Marini mixed finite element discretization of the Darcy flow with non-homogeneous Neumann boundary condition on 2D curved domains. The discretization is defined on a body-fitted triangular mesh, i.e. the boundary nodes of the mesh lie on the curved physical boundary. However, the boundary edges of the triangular mesh, which are straight, may not coincide with the curved physical boundary. A boundary value correction technique is then designed to transform the Neumann boundary condition from the physical boundary to the boundary of the triangular mesh. One advantage of the boundary value correction method is that it avoids using curved mesh elements and thus reduces the complexity of implementation. We prove that the proposed method reaches optimal convergence for arbitrary order discretizations. Supporting numerical results are presented.

AMS subject classifications: 65N15, 65N30

Key words: Mixed finite element method, Neumann boundary condition, curved domain, boundary value correction method.

1 Introduction

Many practical problems arising in science and engineering are posed on the curved domain Ω . Classical finite element methods defined on a polygonal approximation domain Ω_h often suffer from an additional geometric error due to the difference between Ω and Ω_h . This leads to a loss of accuracy for higher-order discretizations [30, 31]. Moreover, when the problem is equipped with an essential boundary condition, effective ways to transfer the boundary condition from $\partial\Omega$ to $\partial\Omega_h$ must be designed. Different strategies

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have been proposed to solve this problem, for example, the isoparametric finite element method [20, 23] and the isogeometric analysis [17, 21]. Another strategy is the boundary value correction method [8], which shifts the essential boundary condition from $\partial\Omega$ to $\partial\Omega_h$, and results in a modified variational formulation. One advantage of the boundary value correction method is that it avoids using curved mesh elements and thus reduces the complexity of implementation. Recently, there are many works utilizing the boundary correction strategy, including the discontinuous Galerkin method via extensions from subdomains [15, 16], transferring technique based on the path integral [27], the shifted boundary method [1, 25], the cutFEM [12], the boundary-corrected virtual element method [2] and the boundary-corrected weak Galerkin method [24], etc.

For the mixed finite element method (MFEM), the Neumann boundary condition becomes essential. Again, higher-order MFEM suffers from a loss of accuracy on curved domains. In [3–5], the authors studied a parametric Raviart-Thomas MFEM on curved domains, which is a generalization of the isoparametric method to the MFEM. In [28], a cutFEM method is proposed in the curved domain, which is based on a primal mixed formulation [7]. The authors of [28] show that the cutFEM method reaches suboptimal convergence.

In this paper, we propose a new boundary corrected MFEM based on the primal mixed formulation [7, 22], and prove that it has optimal convergence rate. Similarly to [18], we weakly impose the Neumann boundary condition. Then, a boundary value correction technique is designed to pull the Neumann boundary data from $\partial\Omega$ to $\partial\Omega_h$. However, unlike the boundary correction in [1, 8, 25], which considers the Dirichlet boundary condition, the Neumann boundary condition involves the outward normal vector on the boundary, which is different in $\partial\Omega$ and $\partial\Omega_h$. This poses extra difficulty in the design and analysis of our scheme. Finally, following [14], we added a term $(\operatorname{div}\cdot, \operatorname{div}\cdot)$ to the discrete formulation to increase stability.

The paper is organized as follows. In Section 2, we introduce some notation and settings. In Section 3, we describe the model problem and introduce the boundary value correction method. In Section 4, the discrete space and the discrete form are defined and analyzed. In Section 5, we prove the optimal error estimate. In Section 6, numerical results are presented. Finally, we draw our conclusions in Section 7.

2 Notations and preliminaries

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz continuous and piecewise C^2 boundary Γ . (The extension to $\mathbb{R}^3$ is possible, but requires some non-trivial technical details.) Let $\mathcal{T}_h$ be a body-fitted triangulation of Ω , i.e., all boundary vertices of $\mathcal{T}_h$ lie on Γ . We assume that $\mathcal{T}_h$ is geometrically conforming, shape regular, and quasi-uniform. Denote by Ω_h the polygonal region occupied by the triangular mesh $\mathcal{T}_h$, and by Γ_h the boundary of Ω_h . When Ω has a curved boundary, the polygonal region Ω_h does not coincide with Ω , as shown in Fig. 1. Denote by $\mathcal{E}_h^b$ the set of all boundary edges in $\mathcal{T}_h$ and by $\mathcal{T}_h^b$ all