

Data Generation-Based Operator Learning for Solving Partial Differential Equations on Unbounded Domains

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Abstract. Wave propagation problems are typically formulated as partial differential equations (PDEs) on unbounded domains to be solved. The classical approach to solving such problems involves truncating them to problems on bounded domains by designing the artificial boundary conditions or perfectly matched layers, which typically require significant effort, and the presence of nonlinearity in the equation makes such designs even more challenging. Emerging deep learning-based methods for solving PDEs, with the physics-informed neural networks (PINNs) method as a representative, still face significant challenges when directly used to solve PDEs on unbounded domains. Calculations performed in a bounded domain of interest without imposing boundary constraints can lead to a lack of unique solutions thus causing the failure of PINNs. In light of this, this paper proposes a novel and effective data generation-based operator learning method for solving PDEs on unbounded domains. The key idea behind this method is to generate high-quality training data. Specifically, we construct a family of approximate analytical solutions to the target PDE based on its initial condition and source term. Then, using these constructed data comprising exact solutions, initial conditions, and source terms, we train an operator learning model called MIONet, which is capable of handling multiple inputs, to learn the mapping from the initial condition and source term to the PDE solution on a bounded domain of interest. Finally, we utilize the generalization ability of this model to predict the solution of the target PDE. The effectiveness of this method is exemplified by solving the wave equation and the Schrödinger equation defined on unbounded domains. More importantly, the proposed method can deal with nonlinear problems, which has been demonstrated by solving Burgers' equation and Korteweg-de Vries (KdV) equation. The code is available at <https://github.com/ZJLAB-AMMI/DGOL>.

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1 Introduction

Real-world wave propagation problems in various fields, such as acoustics, aerodynamics, solid geophysics, oceanography, meteorology, and electromagnetics, are commonly described by partial differential equations (PDEs) on unbounded (or very large) domains. However, solving these PDEs numerically poses challenges due to the infinite domains involved. Standard domain-based numerical methods like finite difference and finite element methods are not directly applicable to solving such unbounded PDEs. The main reason is that these methods rely on discretizing the domain into a finite set of points or elements, leading to a reduced algebraic system with a finite number of degrees of freedom. However, when dealing with unbounded domains, these approaches lead to an algebraic system with an infinite number of degrees of freedom that cannot be effectively solved.

Two popular techniques to deal with the problems on unbounded domains are the artificial boundary method (ABM) [1, 10, 11] and perfectly matched layer (PML) [4, 23]. The ABM involves designing suitable absorbing/artificial boundary conditions (ABCs) that are satisfied by the solution of the original problem on the artificial boundaries. This approach reduces the original unbounded problem to a well-posed boundary value problem on the bounded computational domains of interest. The key ingredient of ABM is the construction of the ABCs. Based on the Fourier series expansion, Laplace transform, z -transform, Padé approximation, continued fraction expansion, and other techniques, exact or approximated ABCs are designed for various linear PDEs. However, it remains challenging to construct suitable ABCs for many nonlinear equations, even for some fundamental equations like the nonlinear Schrödinger equation [11, 34]. The PML method is used as absorbing layers that effectively eliminate reflections for all incident waves, regardless of their frequency and angle. PMLs have gained widespread usage due to their computational efficiency, ease of implementation, applicability to complex geometries, and high absorption accuracy. However, a major drawback of the PML method is its numerical instabilities in time-domain simulations for some wave propagation problems in anisotropic and/or dispersive media [2, 3, 18, 33]. Several methods have been proposed to avoid and remove growing waves and to improve the stability and accuracy of PML formulations. Nevertheless, for some specific situations, such as linear elastodynamic equations in arbitrary anisotropic elastic media, developing a stable PML formulation remains an open problem [23].

Recently, the rapid progress in deep learning has driven the development of solution techniques for PDEs. Learning-based PDE approaches can fall into two categories in terms of the objects approximated by neural networks (NN), i.e., the solution and the