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A Time Variable Filter Approach for the Fluid-Fluid Interaction

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Abstract. This paper analyzes a time variable filter approach for the nonlinear fluid-fluid interaction problem. The method applies a time variable filter to improve numerical solution of a variable time step Euler scheme with explicit second-order extrapolation treatment for nonlinear convection and interface terms. Compared with classical second-order method in time, the proposed approach improves time accuracy from the first order to the second order by adding several lines to the code of variable time step Euler scheme. Theoretically, we prove the unconditional energy stability, local H^1 stability and error estimates. Numerically, some numerical experiments are provided to test the theoretical results, which illustrate the accuracy and efficiency of the presented method.

AMS subject classifications: 65M15, 65M60

Key words: Fluid-fluid interaction, unconditional energy stability, local H^1 stability, time variable filter, error estimate.

1 Introduction

In this paper, we design and analysis a numerical approach of the nonlinear coupled fluid-fluid interaction problem, which is driven by the Gaucker-Manning law [35] for the multi-domain fluid and governed by the following two time-dependent nonlinear

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Navier-Stokes equations [27]: for $i,j=1,2$ and $i\neq j$,

$$\begin{aligned}
\mathbf{u}_{i,t}-\nu_i\Delta\mathbf{u}_i=\mathbf{f}_i-(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i-\nabla p_i \quad \& \quad \nabla\cdot\mathbf{u}_i=0 \qquad & \text{in } \Omega_i,\\
-\nu_i\mathbf{n}_i\cdot\nabla\mathbf{u}_i\cdot\boldsymbol{\tau}=\kappa|\mathbf{u}_i-\mathbf{u}_j|(\mathbf{u}_i-\mathbf{u}_j)\cdot\boldsymbol{\tau} \quad \& \quad \mathbf{u}_i\cdot\mathbf{n}_i=0 \qquad & \text{on } I,\\
\mathbf{u}_i(t)=\mathbf{0} \qquad \text{on } \Gamma_i:=\partial\Omega_i\backslash I, \quad \& \quad \mathbf{u}_i(0,\mathbf{x}_i)=\mathbf{u}_{i,0}(\mathbf{x}_i) \qquad & \text{in } \Omega_i,
\end{aligned} \tag{1.1}$$

for $(\mathbf{x}_i,t)\in\Omega_i\times(0,T]$, $i=1,2$, where $\Omega_i\subset\mathbb{R}^d$ ($d=2$ or 3) are bounded domain coupled across their shared interface I and $T>0$ is final time. Here, $\mathbf{u}_i(\mathbf{x}_i,t):\Omega_i\times(0,T]\rightarrow\mathbb{R}^d$, $p_i(\mathbf{x}_i,t):\Omega_i\times(0,T]\rightarrow\mathbb{R}$ and $\mathbf{f}_i(\mathbf{x}_i,t):\Omega_i\times(0,T]\rightarrow\mathbb{R}^d$ denote the velocity field, the pressure of the flow and the body force on the flow in Ω_i, respectively. Besides, ν_i, $\kappa>0$ and $|\cdot|$ represent the kinematic viscosity, friction coefficient and Euclidean norm. The vector $\mathbf{n}_i$ is the unit normal on $\partial\Omega_i$, and $\boldsymbol{\tau}$ is any vector on I such that $\boldsymbol{\tau}\cdot\mathbf{n}_i=0$. Note that the nonlinear interface and no penetration conditions on I ensure that the momentum flux across the interface is conserved, in spite of the energy is passed between the two domains [35]. This particular property is often used in many oceanography models [29], when one considers that the ocean surface flow is influenced by the surface air, such as climate models, hurricane propagation, coastal weather prediction and so on (see, e.g., [6,30]).

In the past two decades, many numerical methods and mathematical analyses regarding the coupled fluid-fluid interaction problem have been studied and tested. Connors et al. [9] studied a specialized partitioned time stepping method (called the GA method), which decouples the discrete fluid-fluid interaction equations without sacrificing stability and maintaining convergence. Based on unconditional stability of the GA method, several stable schemes have undergone some evolution and been well further developed [1, 3, 8, 19, 20, 22, 23]. Besides, Bresch and Koko [7] have combined optimization-based non-overlapping domain decomposition method with operator-splitting method for this coupled equations. The implicit/exciplict method is firstly proposed in [9] for the fluid-fluid interaction problem and proved to be conditionally stable by Zhang et al. [43, 44]. Unlike the above numerical methods for the fluid-fluid interaction problem, Li et al. [21] have designed a decoupled method which also does not sacrifice the property of unconditional stability and maintains the corresponding convergence with help of auxiliary variable technique.

Concerning the temporal accuracy, Qian et al. [31] have applied the Crank-Nicolson leap-frog scheme to treat the temporal discretization, which improves the time accuracy to 1.5, due to nonlinearity of interface terms. Furthermore, Aggul et al. [2] have developed a predictor-corrector-type method which is unconditionally stable with a second-order time accuracy. By combining the defect-deferred correction method with a sub-grid artificial viscosity type stabilization, an unconditional stability and second-order convergence method is considered for the considered problem with small viscosity [4]. Lately, based on large eddy simulation with correction model, Aggul et al. [5] have proposed a novel turbulence model for the fluid-fluid interaction problem, in which the correction step is used to reduce the modeling error and improve temporal accuracy to 2. However, unfortunately, as stated in [2], the cost of the corrector-type method is around 3 to 4