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Asymptotic-Preserving Discretization of Three-Dimensional Plasma Fluid Models

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Abstract. We elaborate a numerical method for a three-dimensional hydrodynamic multi-species plasma model described by the Euler-Maxwell equations. Our method is inspired by and extends the one-dimensional scheme from [P. Degond, F. Deluzet, and D. Savelief, *Numerical approximation of the Euler-Maxwell model in the quasineutral limit*, Journal of Computational Physics, 231 (4), pp. 1917–1946, 2012]. It can cope with large variations of the Debye length λ_D and is robust in the quasi-neutral limit $\lambda_D \to 0$ thanks to its *asymptotic-preserving* (AP) property. The key ingredients of our approach are (i) a discretization of Maxwell's equations based on primal and dual meshes in the spirit of *discrete exterior calculus* (DEC) also known as the finite integration technique (FIT), (ii) a finite volume method (FVM) for the fluid equations on the dual mesh, (iii) *mixed implicit-explicit timestepping*, (iv) special no-flux and contact boundary conditions at an outer cut-off boundary, and (v) additional *stabilization* in the non-conducting region outside the plasma domain based on direct enforcement of Gauss' law. Numerical tests provide strong evidence confirming the AP property of the proposed method.

AMS subject classifications: 65M15, 65N08, 76X05

Key words: Asymptotic-preserving scheme, finite integration technique, Maxwell-Euler equations.

1 Introduction

The starting point for this work was the desire to numerically simulate the formation and evolution of electric arcs at atmospheric pressures. Following common practise, we rely on a mathematical description by means of a hydrodynamic multi-species plasma model, which boils down to an extended Maxwell-Euler system. The arc phenomenon covers a wide range of plasma regimes and, thus, the design goal was a numerical model

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capable of dealing with all of them seamlessly and simultaneously. Thus, in this work, as in [11], the focus is on *asymptotic-preserving* (AP) discretization in the quasi-neutral limit $\lambda \to 0$, where λ is the rescaled Debye length, see Section 2.2 for the underlying scaling arguments and [9,11] for a discussion of the importance of the AP property of numerical plasma models.

Our approach is inspired by [11], but goes beyond that work in various directions:

- We supplement the Maxwell-Euler system by inter-species friction terms.
- We extend the spatially one-dimensional scheme of [11] to three spatial dimensions using discrete exterior calculus (DEC) to discretize Maxwell's equations combined with a low-order finite volume method for Euler's equations.
- We propose a stabilization in non-conducting regions which is essential for the efficacy of our method.

We stay close to [11] in terms of discretization in time employing semi-implicit time-stepping and base our illustration on the model situations shown in Fig. 1. Yet, we would like to remark that the mesh-based numerical scheme proposed in this paper can in principle be adapted to settings more general than those of Fig. 1. In particular, we do not assume any rotational symmetry, which would allow reduction to two spatial dimensions [35].

The quasi-neutral limit of the Maxwell-Euler system leads to a singularly perturbed problem, that is, the limiting PDE system changes its type[†]. This poses a challenge for simulations in settings encompassing different regimes. Beside the quasi-neutral limit, we remind that singularly perturbed problems occur in extensively many physical models, e.g., in the case of vanishing viscosity problem [3,28,48], the zero-relaxation-limit of kinetic-type equations [1,43], and the incompressible limit of compressible flows [29]. Throughout, it is essential that numerical schemes remain valid even if crucial model parameters approach the limit. These schemes are then said to be *asymptotic-preserving* (AP): Let us assume that we discretize a parameter (denoted by λ) dependent model P^λ, which converges to a limit P^0 as $\lambda \to 0$, by the scheme P_h^λ where h denotes some discretization parameter, e.g., the mesh size. The AP property amounts to uniform convergence of P_h^λ to the P^λ as $\lambda \to 0$. The concept is effectively depicted by a commuting diagram in [27, Fig. 1]. AP schemes have been extensively studied for various problems, e.g., the diffusive limit of kinetic equations [26,34], the low Mach-number limit of compressible flow models [10,19], magnetohydrodynamics [8], the quasi-neutral limit of drift-diffusion equations [2,5]. For more information readers are referred to the comprehensive reviews [9,27].

The content of this paper is organized as follows: In Section 2, we give a full description of the underlying equations of the Maxwell-Euler system and its rescaling procedure. Meanwhile, the boundary conditions in our setting is elaborated. A reformulation

[†]The type of the Maxwell-Euler system switches from hyperbolic to mixed for $\lambda_D \to 0$.