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Exact Solution of the Riemann Problem for the One-Dimensional Blood Flow Equations with General Constant Momentum Correction Coefficient

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Abstract. The momentum correction coefficient in one-dimensional blood flow models is related to the prescribed velocity profile. The exact solution of the Riemann problem for the one-dimensional blood flow equations has been previously studied only for a momentum correction coefficient equal to one (corresponding to a flat velocity profile, i.e. an inviscid fluid). In this paper we solve exactly the Riemann problem for the non-linear hyperbolic one-dimensional blood flow equations with a general constant momentum correction coefficient and a tube law that allows to describe both arteries and veins with continuous or discontinuous mechanical and geometrical properties. In the case of discontinuous properties, only the subsonic regime is considered. We propose a numerical procedure to compute the obtained exact solution and finally we validate it numerically, by comparing exact solutions to those obtained with well-known, first order, numerical schemes on a carefully designed set of test problems. A detailed knowledge about this problem will allow to determine coupling and boundary conditions arising when these models are applied on networks of vessels, ensuring full consistency with the underlying one-dimensional blood flow model without resorting to linearization techniques commonly applied when the momentum correction coefficient is assumed to be different from one.

AMS subject classifications: 76Z05, 35L45, 65M08

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1 Introduction

One-dimensional (1D) blood flow models [13] have been extensively employed for investigating wave propagation phenomena in arteries and veins within a single vessel or in vascular networks [9, 19–21, 30, 38]. These models offer significantly reduced computational complexity compared to full three-dimensional (3D) Fluid Structure Interaction (FSI) models [42], rendering them an appealing option for studying wave propagation phenomena when computational efficiency is a priority.

The 1D blood flow equations are a non-linear system of two partial differential equations with source term. These equations represent conservation of mass and balance of momentum and are obtained by cross-sectional averaging the 3D Navier-Stokes equations, and including a tube law describing the interaction between vessel wall and fluid [10, 13]. The space- and time-dependent unknowns are the cross-sectional area, flow rate and pressure. To close the system, elastic tube laws have been proposed, which distinguish between arteries and veins. In Bernard et al. [1], 1D blood flow equations were derived for the first time from the 3D incompressible Navier-Stokes equations. In more recent times researchers have addressed the description of flow in collapsible tubes (for comprehensive reviews see [10, 26] and the many references therein). During the derivation of the 1D blood flow equations the velocity profile of the axial component of velocity has to be prescribed. This in turn determines a couple of coefficients appearing in the momentum balance equation, one related to dissipation due to friction between blood and the vessel wall and a second one, called momentum correction coefficient, which is related to the convective term of the equation. In this paper the momentum correction coefficient will be represented by greek letter α.

1D blood flow models with discontinuous properties are representative of situations in which some geometrical and mechanical properties that characterize compliant vessels change rapidly in space, for example due to the insertion of stents (expandable metal meshes) in arteries or in veins after a surgical procedure with the purpose of returning the vessel lumen to approximately its original shape. This causes an abrupt variation in the elastic properties of the vessel wall, since the stent is usually different from the soft arterial tissue [29, 41]

From both the mathematical and numerical points of view, a basic problem to be solved is the special Cauchy problem called the Riemann problem [12, 16, 36]. This is of paramount importance because it provides an analytical solution to evaluate the performance of the numerical methods. For arteries, the exact solution of the Riemann problem for the 1D blood flow equations is presented in [28, 33, 39]. Veins, compared to arteries, are highly non-linear and exhibit large deformations, including collapse. For the exact solution of the Riemann problem for the 1D blood flow equations in veins see [33, 34, 40]. All these available exact solutions were obtained for a momentum correction coefficient $\alpha = 1$. The complete knowledge of the model eigenstructure and wave relations for $\alpha \neq 1$ is highly desirable since these relations play a fundamental role in the determination of coupling and boundary conditions when 1D blood flow models are applied to networks of