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Derivative-Based Finite-Volume MR-HWENO Scheme for Steady-State Problems

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Abstract. In this paper, we further extend the derivative-based finite-volume multi-resolution Hermite weighted essentially non-oscillatory (MR-HWENO) scheme proposed in our previous article (Li, Shu and Qiu, J. Comput. Phys., 446:110653, 2021) to simulate the steady-state problem. When dealing with the steady-state problem, the process of updating and reconstructing the function values is similar to the previous scheme, but the treatment of the derivative values is changed. To be more specific, instead of evolving in time, in the sense of cell averages, the scheme uses the derivative at the current time step and the function at the next time step to reconstruct the derivative at the next time step by direct linear interpolation. There are two advantages for this approach: the first is its high efficiency, when handling the derivative, neither the update on time nor the calculation of nonlinear weights is required; in the meantime, the CFL number can still be taken up to 0.6 as in the original scheme; the second is its strong convergence, the corresponding average residual can quickly converge to machine accuracy, thus obtaining the desired steady-state solution. One- and two-dimensional numerical experiments are given to verify the high efficiency and strong convergence of the proposed MR-HWENO scheme for the steady-state problems.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Finite-volume, multi-resolution scheme, HWENO scheme, Runge-Kutta method, steady-state problem.

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1 Introduction

In this paper, we apply the derivative-based finite-volume multi-resolution Hermite weighted essentially non-oscillatory (MR-HWENO) scheme [22] to solve the following steady-state problem:

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = s(\boldsymbol{U},\boldsymbol{X}), \tag{1.1}$$

where $\boldsymbol{U}$ is the unknown variable to be determined, $\boldsymbol{F}(\boldsymbol{U})$ is the (usually nonlinear) flux function, $s(\boldsymbol{U},\boldsymbol{X})$ is the given source term and $\boldsymbol{X} = (x_1,\cdots,x_d)$. Only one- and two-dimensional cases are considered in this paper, i.e. d=1 or 2, accordingly, we use x to denote x_1, and y to denote x_2.

The steady-state problem is an important mathematical model, which is widely used in a variety of fields such as compressible fluid dynamics, wave motion, advective transport of matter and so on. However, it is not easy to obtain the solution of these problems either theoretically or numerically. One way to solve Eq. (1.1) numerically is to solve the corresponding unsteady hyperbolic balance law by adopting an appropriate time marching method

$$\begin{cases} \boldsymbol{U}_t + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = s(\boldsymbol{U},\boldsymbol{X}), \\ \boldsymbol{U}(\boldsymbol{X},0) = \boldsymbol{U}_0(\boldsymbol{X}). \end{cases} \tag{1.2}$$

With the advance of time, when the residual of the above unsteady hyperbolic balance law (1.2) is sufficiently small, the corresponding solution is considered acceptable as the steady-state solution of (1.1). In this way, we transform the steady-state problem into a time-dependent hyperbolic balance law problem. But in this case, especially for those equations with a nonlinear flux function, discontinuities may appear even if the initial condition is smooth enough. To address this issue, there have existed many works devoted to designing efficient numerical methods to solve these problems with strong shocks or contact discontinuities. A partial list includes essentially non-oscillatory (ENO) schemes in [19, 35, 36], weighted ENO (WENO) schemes in [21, 28] (see also [8]) and Hermite WENO (HWENO) schemes in [30, 31, 40, 43–45].

When solving the steady-state problem using the classical WENO schemes [21, 28] with an appropriate time discretization method [11], we must address the problem that slight post-shock oscillations may propagate downward from the region near the shock to the smooth region, causing the residual to hang at a high truncation error level rather than to stabilize to machine accuracy, see [42]. Although reconstructing the numerical flux using a limiter or an upwind-biased interpolation technique can improve the convergence of the numerical solution to steady-state as shown in some later papers [33, 41], the residual still fails to converge to machine accuracy for many two-dimensional test cases. For steady-state simulations of Euler equations, Wan and Xia proposed a new hybrid strategy for the fifth-order WENO scheme in [37], which performs better in steady-state convergence property with less dissipative and dispersive errors compared with