Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0002

Vol. **36**, No. 4, pp. 943-976
October 2024

MHDnet: Physics-Preserving Learning for Solving Magnetohydrodynamics Problems

Xiaofei Guan[1], Boya Hu[1], Shipeng Mao[2,3,*], Xintong Wang[1] and Zihao Yang[4,5]

[1] *School of Mathematical Sciences, and Key Laboratory of Intelligent Computing and Applications (Ministry of Education), Tongji University, Shanghai 200092, China.*
[2] *LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.*
[3] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.*
[4] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China.*
[5] *Institute of Mathematics, Henan Academy of Sciences, Zhengzhou 450046, China.*

Received 4 January 2024; Accepted (in revised version) 7 May 2024

Abstract. Designing efficient and high-accuracy numerical methods for complex dynamic incompressible Magnetohydrodynamics (MHD) equations remains a challenging problem in various analysis and design tasks. This is mainly due to the nonlinear coupling of the magnetic and velocity fields occurring with convection and Lorentz forces, and multiple physical constraints, which will lead to the limitations of numerical computation. In this paper, we develop the MHDnet as a physics-preserving learning approach to solve MHD problems, where three different mathematical formulations are considered and named $B\ formulation$, $A_1\ formulation$, and $A_2\ formulation$. Then the formulations are embedded into the MHDnet that can preserve the underlying physical properties and divergence-free condition. Moreover, MHDnet is designed by the multi-modes feature merging with multiscale neural network architecture, which can accelerate the convergence of the neural networks (NN) by alleviating the interaction of magnetic fluid coupling across different frequency modes. Furthermore, the pressure fields of three formulations, as the hidden state, can be obtained without extra data and computational cost. Several numerical experiments are presented to demonstrate the performance of the proposed MHDnet compared with different NN architectures and numerical formulations. In future work, we will develop possible applications of inverse problems for coupled equation systems based on the framework proposed in this paper.

AMS subject classifications: 76-10, 76W05, 76D05, 68T07

*Corresponding author. *Email addresses:* `guanxf@tongji.edu.cn` (X. Guan), `maosp@lsec.cc.ac.cn` (S. Mao), `yangzihao@nwpu.edu.cn` (Z. Yang)

Key words: Magnetohydrodynamics, multiscale neural network, physics-preserving formulation, divergence-free, multi-modes feature.

1 Introduction

The MHD system can be formulated by the fully nonlinear coupled equations including Maxwell equations of electromagnetism and the Navier-Stokes equations of fluid dynamics, which describe the macroscopic interaction of conductive fluids under electromagnetic induction. Here, the MHD equations have been considered in terms of two fundamental effects. First, the motion of conducting materials in the presence of a magnetic field induces an electric current and changes the existing electromagnetic field. Second, the current and magnetic fields generate the Lorentz force, which accelerates fluid particles in a direction perpendicular to the magnetic and current fields. In this paper, we are concerned with the dynamic incompressible MHD equations which have received considerable attention for their important applications in liquid metal modeling and plasma physics, such as metallurgical engineering, electromagnetic pumping, the stirring of liquid metals, liquid metal cooling of nuclear reactors and induction-based flow measurement [1–5].

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d=2,3)$ with connected boundary $\partial\Omega$, and the following time-dependent incompressible MHD equations are considered:

$$\boldsymbol{u}_t - R_e^{-1}\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \boldsymbol{J}\times\boldsymbol{B} = \boldsymbol{f}, \quad \text{in } Q_T, \tag{1.1}$$

$$\boldsymbol{B}_t + \operatorname{curl}\boldsymbol{E} = 0, \quad \text{in } Q_T, \tag{1.2}$$

$$\boldsymbol{J} = \kappa(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}), \quad \text{in } Q_T, \tag{1.3}$$

$$\mu^{-1}\operatorname{curl}\boldsymbol{B} = \boldsymbol{J}, \quad \text{in } Q_T, \tag{1.4}$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \operatorname{div}\boldsymbol{B} = 0, \quad \text{in } Q_T, \tag{1.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \boldsymbol{B}(0) = \boldsymbol{B}^0, \quad \text{in } \Omega, \tag{1.6}$$

where $Q_T = \Omega\times(0,T)$, $T>0$ is a given finite final time, $\boldsymbol{u}$ denotes the velocity field, p is the pressure, $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{J}$ is the electric current density, $\boldsymbol{E}$ is the electric field, R_e is the hydrodynamic Reynolds number, κ is the electric conductivity, μ is the magnetic permeability, and $\boldsymbol{f}$ is a given forcing term. To make the system of equations well-posed, the no-slip and perfectly conducting wall conditions are defined as

$$\boldsymbol{u} = 0, \quad \boldsymbol{n}\times\boldsymbol{B} = 0, \quad \text{in } \partial\Omega\times(0,T). \tag{1.7}$$

In the last several decades, various numerical methods for incompressible MHD problems have been extensively developed in the literature. Let us review the references and try to summarize them, which is unlikely to be complete and accurate, of course. The most commonly used mathematical model for incompressible MHD system Eqs. (1.1)