

A Splitting Method for Nonlinear Filtering Problems with Diffusive and Point Process Observations

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Abstract. This paper aims to develop and analyze a comprehensive discretized splitting-up numerical scheme for the Zakai equation. This equation arises from a nonlinear filtering problem, where observations incorporate noise modeled by point processes and Wiener processes. Initially, we introduce a regularization parameter and employ a splitting-up approach to break down the Zakai equation into two stochastic differential equations and a partial differential equation (PDE). Subsequently, we employ a finite difference scheme for the temporal dimension and the spectral Galerkin method for the spatial dimension to achieve full discretization of these equations. This results in a numerical solution for the Zakai equation using the splitting-up technique. We demonstrate that this numerical solution converges to the exact solution with a convergence order of $\frac{1}{2}$. Additionally, we conduct several numerical experiments to illustrate and validate our theoretical findings.

AMS subject classifications: 35K25, 35K55, 35K91, 65M12, 65M15

Key words: Nonlinear filtering problem, Zakai equation, splitting-up technique, error analysis.

1 Introduction

The objective of the filtering problem is to seek an optimal estimate of the unobserved state for a stochastic dynamical system with partial observations. Nonlinear filtering models find extensive application in various fields, such as biology [17], physics [27], target tracking [32], and weather forecasting [2], among others. When modeling the noise

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in the observations of a filtering problem, the Wiener process noise, which is a continuous process, is routinely assumed due to its ubiquity and convenience. However, in many applications, such as modeling the number of customers arriving at a supermarket [16] or the number of births in a given period of time [33], it is more appropriate to assume that the observations include both the Wiener process and the Poisson process.

Several notable achievements have been made in the study of nonlinear filtering models driven by Lévy processes. In [29], Qiao and Duan investigated a nonlinear filtering model where Lévy processes drive both the signal and observation. They derived a Zakai equation and a Kushner-Stratonovich equation, demonstrating the existence and uniqueness of strong solutions, respectively. Fernando and Hausenblas [19] studied a nonlinear filtering model featuring correlated Poisson noise. They also derived a Zakai equation and established sufficient conditions to ensure the existence and uniqueness of the solution. In [14] Ceci and Colaneri directed their attention to a nonlinear filtering model where the dynamics are modeled by correlated diffusion and Poisson processes. They derived an associated Zakai equation and proved the pathwise uniqueness of solutions. In another study, Frey et al. [22] applied a spectral Galerkin method to a Zakai equation stemming from a nonlinear filtering model involving the Poisson process. They successfully demonstrated the convergence properties of the spatial semi-discretized approximate solution.

The primary objective of this research is to develop an efficient numerical method for approximating the Zakai equation in nonlinear filtering problems where observations involve noises characterized by point processes and Wiener processes. This method combines the splitting-up technique, finite difference methodology, and spectral Galerkin approach. We aim to provide a comprehensive error analysis of this numerical method. Splitting-up methods have wide applications across a variety of fields, as documented in the literature [7, 11–13, 26]. These techniques can transform complex equations into several more easily solvable components, thereby enhancing computational efficiency. Notably, splitting-up methods have been successfully employed to approximate the Zakai equation within the context of nonlinear filtering models driven by independent or correlated Wiener processes, as evidenced by previous studies [11, 20, 24, 25, 34]. Frey et al. [22], for instance, conducted a comparative analysis of the splitting-up method and the Euler-Maruyama method, highlighting the former's superior accuracy and stability, particularly in scenarios with larger time step sizes. Furthermore, Luo et al. [25] conducted experiments that provide compelling evidence of the particle filter based on the splitting-up method. Their results indicate that this approach delivers both greater accuracy and efficiency when compared to the sampling importance and resampling filter. Bao et al. [6] design an effective algorithm to approximate the high-dimensional nonlinear filtering problems based on the combination of splitting-up finite difference scheme and hierarchical sparse grid method. A practical scenario is that the state dynamics are driven by Lévy process [4, 5, 8, 9]. By transforming the Zakai equation into a backward SDE, Bao et al. [5] proposed a novel numerical approximation to such model and applied which to material science. However, a nonlinear filtering model with jump observations