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Numerical Modeling of Flocking Dynamics with Topological Interactions

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Abstract. In this paper, we propose a numerical investigation of topological interactions in flocking dynamics. Starting from a microscopic description of the phenomena, mesoscopic and macroscopic models have been previously derived under specific assumptions. We explore the role of topological interactions by describing the convergence speed to consensus in both microscopic and macroscopic dynamics, considering different forms of topological interactions. Additionally, we compare mesoscopic and macroscopic dynamics for monokinetic and non-monokinetic initial data. Finally, we illustrate with some simulations in one- and two-dimensional domains the sensitive dependence of solutions on initial conditions, including the case where the system exhibits two solutions starting with the same initial data.

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1 Introduction

Collective dynamics have been widely investigated over the last years in different fields of research, due to the great variety of living and non-living systems exhibiting such complex behaviors. From flocks of birds, schools of fish to bacterial colonies and robotic swarms, coordinated movements of individuals interacting within a group give raise to the so called *emergent behaviors*. Understanding the underlying principles of self-organized behaviors has emerged as a fundamental challenge in applied mathematics,

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aiming at providing the framework to describe, predict, and analyze the basic mechanisms determining the observed collective dynamics. A crucial aspect of mathematical modeling is selecting the appropriate scale for the description for the phenomena under investigation. In traditional microscopic, agent-based approaches, the dynamics of each individual in the group are tracked. While this approach allows to highly represent the interactions between agents, it also leads to increasing computational cost as the number of agents grows. For large flocks, mesoscopic and macroscopic models are generally more efficient. When the number of agents increases, mesoscopic and macroscopic models can be considered as efficient alternatives to pure microscopic ones. This is mainly due to the fact that microscopic models, focusing on the dynamics of each individual, become computationally demanding for large groups. Conversely, mesoscopic and macroscopic approaches allow to describe collective behaviors using aggregated variables such as probabilistic distribution of agents, macroscopic density and average velocity field (see [25] for an updated reference paper on the derivation from microscopic to macroscopic models). Several mathematical models have been proposed in the literature to model *flocking* behaviors, see [33] for an introduction to the field. Typically displayed by flocks of birds, flocking is used to define the coordinated motion of agents reaching the same velocity when moving towards the same direction. The majority of the models, starting from the seminal papers [12, 13], assumes interactions involving metric quantities (*metric interactions*), where individuals adjust their movement in response to the other members of the group, and the interaction depends on the relative Euclidean distance. However, increasing empirical evidence suggests that in many biological systems, interactions are *topological*, meaning that individuals interact with a fixed number of neighbors, regardless of their absolute distances [3, 6]. As a consequence, several mathematical models adopting this novel paradigm have appeared, ranging from microscopic to macroscopic ones, see e.g. [2, 7, 10, 17, 18, 28]. This physically non-metric nature of interactions plays a crucial role for the stability, robustness and efficiency of the observed motion, affecting not only the local organization of individuals but also the emergent macroscopic patterns.

1.1 Topological interactions models: From microscopic to macroscopic scale

In this paper we focus on the topological model of the Cucker-Smale type firstly proposed in [17]. Starting from the microscopic scale, the model reads

$$\begin{cases} \dot{X}_i(t)=V_i(t), \\ \dot{V}_i(t)=\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} p_{i,j}(V_j(t)-V_i(t)), \end{cases} \qquad i\in\{1,\cdots,N\}. \tag{1.1}$$

Here $X_i\in\mathbb{R}^d$, $V_i\in\mathbb{R}^d$ denote positions and velocities of N indistinguishable interacting agents in a general $d\geq 1$ dimensional space. A key factor of the model is the so-called communication weights $p_{i,j}$, modeling the kind of considered interactions.