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An Efficient Time-Splitting Method to Simulate the Dynamics of Spin-Orbit Coupled Spin-1 Bose-Einstein Condensates

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Abstract. In this paper, an efficient time-splitting Fourier spectral method is proposed to simulate the dynamics of spin-orbit coupled spin-1 Bose-Einstein condensates (SOC spin-1 BECs). We split the Hamiltonian into a linear part, which consists of the Laplace and SOC terms, and a nonlinear part that includes all the remaining terms. The linear subproblem is integrated analytically in phase space by solving an ordinary differential system of constant coefficient matrix. While, for the nonlinear subproblem, it is proved the coefficient matrix is actually time-independent in physical space, therefore, the nonlinear subproblem can be integrated exactly. Based on such two-step splitting, we construct high-order schemes to simulate the dynamics. Our method is spectrally accurate in space and high order in time. It is efficient, explicit, unconditionally stable and simple to implement. In addition, we derive some dynamical properties for SOC spin-1 BECs. Extensive numerical results are presented to confirm the accuracy and efficiency, illustrate the dynamical properties at discrete level, and show interesting physics of SOC spin-1 BECs, such as the SOC effects and different wave patterns.

AMS subject classifications: 35Q41, 65M70, 81Q05, 81V45

Key words: spin-1 Bose Einstein condensate, spin-orbit coupled, Gross-Pitaevskii equation, dynamics, fast Fourier transform, time-splitting method.

1 Introduction

Since its experimental realizations in 1995 [1, 11], the Bose-Einstein condensate (BEC) has stimulated great excitement in the physical community and regains vast interests

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in atomic and molecular physics as well as condensate matter physics. In particular, the spin-orbit coupling (SOC), which was found that plays a crucial role in Majorana fermions [27], spintronic devices [18], spin Hall effect [15] and topological insulators [14], has been successfully induced in recent experiments in a neutral atomic BECs by dressing two atomic spin states with a pair of lasers [20–22]. These experiments triggered a strong activity in the area of spin-orbit-coupled cold atoms and a number of exciting phenomena have been discovered. The spin-1 BECs with isotropic spin-orbit coupling and rotation have been also studied [19,31].

It is well known that, in the mean field regime, the spin-$\mathcal{F}$ ($\mathcal{F}\in\mathbb{N}$) BEC can be well described by a system of $2\mathcal{F}+1$ coupled Gross-Pitaevskii equations (GPEs) when the temperature T is much smaller than the critical temperature T_c. Thus the spin-orbit coupled spin-1 BEC can be described by the macroscopic complex-valued vector wave function $\Psi=\Psi(\mathbf{x},t)=(\psi_1(\mathbf{x},t),\psi_0(\mathbf{x},t),\psi_{-1}(\mathbf{x},t))^T$ satisfying the GPEs

$$
\begin{aligned}
i\hbar\partial_t\psi_1(\mathbf{x},t)&=\left[-\frac{\hbar^2}{2m}\nabla^2+V+\beta_0\rho+\beta_1F_z\right]\psi_1+\frac{\beta_1}{\sqrt{2}}F_-\psi_0-\gamma\widetilde{L}_0\psi_0,\\
i\hbar\partial_t\psi_0(\mathbf{x},t)&=\left[-\frac{\hbar^2}{2m}+V+\beta_0\rho\right]\psi_0+\frac{\beta_1}{\sqrt{2}}[F_+\psi_1+F_-\psi_{-1}]-\gamma(\widetilde{L}_0\psi_{-1}+\widetilde{L}_1\psi_1),\\
i\hbar\partial_t\psi_{-1}(\mathbf{x},t)&=\left[-\frac{\hbar^2}{2m}\nabla^2+V+\beta_0\rho-\beta_1F_z\right]\psi_{-1}+\frac{\beta_1}{\sqrt{2}}F_+\psi_0-\gamma\widetilde{L}_1\psi_0,\\
\psi_\ell(\mathbf{x},0)&=\psi_\ell^0(\mathbf{x}),\quad \ell=1,0,-1,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{x}=(x,y,z)^\top\in\mathbb{R}^3$ is the Cartesian coordinate vector, t is time, $\hbar$ is the Planck constant, m is the atomic mass. $\beta_0=\frac{4\pi\hbar^2}{3m}(a_0+2a_2)$ and $\beta_1=\frac{4\pi\hbar^2}{3m}(a_2-a_0)$ are constants expressed in terms of the s-wave scattering lengths a_0 and a_2 for a scattering channel of total hyperfine spin 0 (antiparallel spin collision) and spin 2 (parallel spin collision), respectively. $\rho=\rho_1+\rho_0+\rho_{-1}$ is the total density with $\rho_\ell=|\psi_\ell|^2$ ($\ell=1,0,-1$) being the density of each spin component. $\widetilde{L}_0=\hbar(i\partial_x+\partial_y)$ and $\widetilde{L}_1=\hbar(i\partial_x-\partial_y)$ are the spin-orbit coupling operators, γ is the spin-orbit coupling strength. $V(\mathbf{x})$ is a given real-valued external trapping potential determined by the type of system under investigation. In most BEC experiments, a harmonic potential is chosen to trap the condensates, i.e.,

$$
V(\mathbf{x})=\frac{1}{2}\left(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2\right),
\tag{1.2}
$$

where ω_v ($v=x,y,z$) are dimensionless constants representing the trapping frequencies in v-direction. The wave function is normalized according to

$$
\|\Psi\|^2:=\int_{\mathbb{R}^3}\sum_{\ell=-1}^{1}|\psi_\ell(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}=N,
$$