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Improved Randomized Neural Network Methods with Boundary Processing for Solving Elliptic Equations

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Abstract. We present two improved randomized neural network methods, namely the RNN-Scaling and RNN-Boundary-Processing (RNN-BP) methods, for solving elliptic equations such as the Poisson equation and the biharmonic equation. The RNN-Scaling method modifies the optimization objective by increasing the weight of boundary equations, resulting in a more accurate approximation. We propose the boundary processing techniques for the rectangular domain that enforce the RNN method to satisfy the non-homogeneous Dirichlet and clamped boundary conditions exactly. We further prove that the RNN-BP method is exact for solutions with specific forms and validate it numerically. Numerical experiments demonstrate that the RNN-BP method is the most accurate among the three methods, with the error reduced by up to 6 orders of magnitude for some tests.

AMS subject classifications: 65M08, 35R05

Key words: Randomized neural network, elliptic equations, boundary conditions, scaling method.

1 Introduction

Elliptic partial differential equations (PDEs) model the steady-state conditions in various physical phenomena, including electrostatics, gravitational fields, elasticity, phase-field models, and image processing [1, 10, 30]. For instance, the Poisson equation characterizes the distribution of a scalar field based on boundary conditions and interior sources. In contrast, the biharmonic equation is employed to model phenomena such as the deflection of elastic plates and the flow of incompressible, inviscid fluids. Solving these

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equations is essential for understanding and predicting system behavior in various applications. Traditional numerical methods for solving elliptic equations, such as finite difference methods [2,3,21,32], finite element methods [6,19,20,22,23,34,35], finite volume methods [14,26,27] and spectral methods [4,7,18] have been well studied and widely used. However, these methods often require careful discretization to obtain numerical solutions with high accuracy. Moreover, they may face challenges in handling mesh generation on complex domains and boundary conditions.

In recent years, deep neural network (DNN) methods have been greatly developed in various fields, such as image recognition, natural language processing, and scientific computing. One area where DNN has shown promise is in solving PDEs, including elliptic equations. The DNN-based method transforms the process of solving PDEs into optimization problems and utilizes gradient backpropagation to adjust the network parameters and minimize the residual error of the PDEs. Several effective DNN-based methods include the Physics-Informed Neural Networks (PINNs) [24], the deep Galerkin method [28], the deep Ritz method (DRM) [9], and the deep mixed residual method [17], among others [8,33]. The main difference between these methods lies in the construction of the loss function.

PINNs offer a promising approach for solving various types of PDEs. However, they still have limitations. One major limitation is the relatively low accuracy of the solutions [11], the absolute error rarely goes below the level of 10^{-3} to 10^{-4}. Accuracy at such levels is less than satisfactory for scientific computing, and in some cases, they may fail to converge. Another limitation is that PINNs require high computational cost and training time, which makes them less practical for large-scale or complex problems. PINNs require substantial resources to integrate the PDEs into the training process, especially for the problems involving high-dimensional PDEs or those requiring fine spatial and temporal resolutions.

RNN has recently attracted increasing attention for its application in solving partial differential equations. The weights and biases of the RNN method are randomly generated and fixed, and do not need to be trained. The optimization problem of PINNs is usually a complicated nonlinear optimization problem, requiring a great number of training steps. For the RNN method, the resulting optimization problem is a least-squares problem, which can be solved without training steps.

For deep neural networks, the exact imposition of boundary and initial conditions is crucial for the training speed and accuracy of the model, since it may accelerate the convergence of the training process and improve overall accuracy. For instance, the inexact enforcement of boundary and initial conditions severely affects the convergence and accuracy of PINN-based methods [29]. Recently, many methods have been developed for the exact imposition of Dirichlet and Neumann boundary conditions, which leads to more efficient and accurate training. The main approach is to divide the numerical approximation into two parts: a deterministic function satisfying the boundary condition and a trainable function with the homogeneous condition. This idea was first proposed by Lagaris et al. in [12,13]. The exact enforcement of boundary conditions is applied in