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Graph-Based Methods for Hyperbolic Systems of Conservation Laws Using Discontinuous Space Discretizations

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Abstract. We present a graph-based numerical method for solving hyperbolic systems of conservation laws using discontinuous finite elements. This work fills important gaps in the theory as well as practice of graph-based schemes. In particular, four building blocks required for the implementation of flux-limited graph-based methods are developed and tested: a first-order method with mathematical guarantees of robustness; a high-order method based on the entropy viscosity technique; a procedure to compute local bounds; and a convex limiting scheme. Two important features of the current work are the fact that (i) boundary conditions are incorporated into the mathematical theory as well as the implementation of the scheme. For instance, the first-order version of the scheme satisfies pointwise entropy inequalities including boundary effects for any boundary data that is admissible; (ii) sub-cell limiting is built into the convex limiting framework. This is in contrast to the majority of the existing methodologies that consider a single limiter per cell providing no sub-cell limiting capabilities. From a practical point of view, the implementation of graph-based methods is algebraic, meaning that they operate directly on the stencil of the spatial discretization. In principle, these methods do not need to use or invoke loops on cells or faces of the mesh. Finally, we verify convergence rates on various well-known test problems with differing regularity. We propose a simple test in order to verify the implementation of boundary conditions and their convergence rates.

AMS subject classifications: 35L65, 35Q31, 65M12, 65N30, 65M22, 65M60

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1 Introduction

For the last four decades the field of numerical methods for solving hyperbolic systems of conservation equations has been dominated by a paradigm that is commonly referred to as high-resolution schemes. These are numerical methods in which the order of consistency is automatically adjusted locally (in space) depending on some chosen smoothness criteria; see early references [4, 27, 55, 61]. While a heuristic high-resolution method is a good starting point for practical computations, it is not enough to achieve unconditional robustness of the scheme. Here, we define unconditional robustness as the guarantee that the computed update at a given time step remains admissible and maintains crucial physical invariants, such that the resulting state can be used again as input for the next time step update. Modern approaches for constructing robust high-order schemes are based on the following ingredients [8,20,25,32,36,39,40,44–47,50,52,56,57,61,63]:

(a) a reference low-order method with mathematically guarantees of robustness,

(b) a formally high-order method that may or may not guarantee any robustness properties,

(c) a post-processing procedure based on either flux or slope limiting techniques that blends the low-order and high-order solutions.

A particular incarnation of such postprocessing technique is the convex limiting technique that establishes mathematical guarantees for maintaining a (local) invariant-set property [20].

First-order graph-based formulations combined with discontinuous spatial discretizations are only tersely described in [25, Section 4.3], leaving the path towards high-performance high-order graph-based methods underdeveloped. Therefore, the first goal of the present paper is to complete the mathematical theory and discuss computational aspects of invariant-set preserving schemes for the case of discontinuous finite elements in comprehensive detail. In particular, we incorporate boundary conditions into the formulation of the scheme, provide proofs of invariant-set preservation and discrete entropy inequalities including the effects of boundary data.

The second goal of this paper is to lay out the elementary building blocks required to construct a robust high-order scheme with convex limiting, using a graph-based discontinuous Galerkin discretization. This requires the development and testing of three components:

(i) a heuristic high-order method,

(ii) a suitable strategy for constructing local bounds and their relaxation in order to prevent degradation to first-order accuracy,

(iii) a convex-limiting procedure that blends the high and low order methods while maintaining an invariant set.