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Equilibrium Preserving Space in Discontinuous Galerkin Methods for Hyperbolic Balance Laws

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Abstract. In this paper, we develop a general framework for the design of the arbitrary high-order well-balanced discontinuous Galerkin (DG) method for hyperbolic balance laws, including the compressible Euler equations with gravitation and the shallow water equations with horizontal temperature gradients (referred to as the Ripa model). Not only does the hydrostatic equilibrium include the more complicated isobaric steady state in the Ripa system, but our scheme is also well-balanced for the exact preservation of the moving equilibrium state. The strategy adopted is to approximate the equilibrium variables in the DG piecewise polynomial space, rather than the conservative variables, which is pivotal in the well-balanced property. Our approach provides flexibility in combination with any consistent numerical flux, and it is free of the reference equilibrium state recovery and the special source term treatment. This approach enables the construction of a well-balanced method for non-hydrostatic equilibria in Euler equations. Extensive numerical examples such as moving or isobaric equilibria validate the high order accuracy and exact equilibrium preservation for various flows given by hyperbolic balance laws. With a relatively coarse mesh, it is also possible to capture small perturbations at or close to steady flow without numerical oscillations.

AMS subject classifications: 65M60, 35L65, 35Q35

Key words: Euler equations with gravitation, Ripa model, discontinuous Galerkin method, equilibrium preserving space, well-balanced.

1 Introduction

A class of hyperbolic conservation laws with source terms, also known as hyperbolic balance laws, is considered in this paper. They have been widely used in various fields, including chemistry, biology, fluid dynamics, astrophysics, meteorology, etc. In the one-

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dimensional case, such a model takes the form of the following equation:

$$u_t+f(u)_x=r(u), \tag{1.1}$$

where $u,f(u),r(u)$ denote the conservative variable, the physical flux, and the source term, respectively. An important feature of such models is that they usually admit non-trivial steady-state solutions, in which the non-zero flux gradient exactly balances the source term.

As a typical example of hyperbolic balance laws, the shallow water equations incorporating the horizontal temperature gradients were introduced in [41] for modeling ocean currents. We often refer to it as the Ripa model. The shallow water equations are derived from the incompressible Navier-Stokes equations, assuming the density is constant. The Ripa system is a generalized model of the shallow water equations. Based on multi-layered ocean models with several layers of different constant densities, the Ripa system is obtained by vertically averaging each layer's density, horizontal pressure gradient, and velocity fields. The introduction of horizontal temperature gradients is advantageous to represent the variations in the fluid density within each layer.

Due to the presence of the source term, many standard numerical methods may only capture small perturbations of the near-equilibrium flow if extremely refined meshes are used. As a result, the well-balanced schemes were first proposed by Bermudez and Vazquez [2] in the shallow water equations to preserve the exact equilibrium state solutions at a discrete level. There has been tremendous interest in constructing well-balanced schemes for the Ripa system in the last few years. Chertock *et al.* [9] first proposed a well-balanced and positivity-preserving central-upwind scheme using an interface tracking technique. A second-order accurate unstaggered central finite volume scheme was developed by Touma and Klingenberg [45]. Based on the path-conservative approximate Riemann solver, [43] derived an HLLC type scheme for the Ripa model, which possessed the well-balanced and positivity-preserving property at the same time. To achieve high order accuracy as well as capture the small perturbations of the hydrostatic state well without numerical oscillations near the discontinuity, several works [23, 35, 40] under the framework of finite difference and discontinuous Galerkin methods were proposed. They extended the well-balanced approach based on modifying the numerical fluxes with the hydrostatic reconstruction and the special source term splitting used in the shallow water equations to the Ripa model. These schemes above mainly focus on the still water steady state (with zero velocity). Similar to the shallow water equations, there are some limitations in these schemes developed for the still water equilibrium that cannot be generalized to the moving water equilibrium (with non-zero velocity), which is much more difficult. The second-order surface reconstruction schemes were introduced in [14] to maintain the still water and moving water equilibrium state solutions and guarantee the positivity of the water height and the temperature. The well-balanced discontinuous Galerkin method, which decomposed the numerical results into the equilibrium and perturbation part, was first introduced by [49] for arbitrary equilibria of the shallow water equations. Britton and Xing [4] extended the analogous idea to the Ripa model with