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Dirichlet-Neumann Learning Algorithm for Solving Elliptic Interface Problems

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Abstract. Non-overlapping domain decomposition methods are well-suited for addressing interface problems across various disciplines, where traditional numerical simulations often require the use of interface-fitted meshes or technically designed basis functions. To remove the burden of mesh generation and to effectively tackle with the flux transmission condition, a novel mesh-free scheme, i.e., the Dirichlet-Neumann learning algorithm, is studied in this work for solving the benchmark elliptic interface problems with high-contrast coefficients and irregular interfaces. By resorting to the variational principle, we carry out a rigorous error analysis to evaluate the discrepancy caused by the boundary penalty treatment for each decomposed subproblem, which paves the way for realizing the Dirichlet-Neumann algorithm using neural network extension operators. Through experimental validation on a series of testing problems in two and three dimensions, our methods demonstrate superior performance over other alternatives even in scenarios with inaccurate flux predictions at the interface.

AMS subject classifications: 65N55, 35J20, 68Q32

Key words: Elliptic interface problem, high-contrast coefficient, compensated deep Ritz method, artificial neural networks, deep learning.

1 Introduction

Many problems in science and engineering are carried out with domains separated by curves or surfaces, e.g., the abrupt change in material properties between adjacent regions, from which the interface problem naturally arises. A widely studied benchmark example is the elliptic interface problem with high-contrast coefficients [6,36,40], whose solution lies in the Sobolev space $H^{1+\epsilon}(\Omega)$ with $\epsilon>0$ possibly close to zero [44]. Due to the

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limited regularity of solution, classical numerical methods, such as finite difference and finite element methods [3, 34], require the generation of an interface-fitted mesh for the discretization of domain [6], which can be technically involved and time consuming especially when the geometry of interface gets complicated and the dimension of problem increases. To ease the burden of mesh generation, numerical approaches based on unfitted meshes, e.g., the immersed interface method [36] and many others, have emerged as attractive alternatives [40]. However, using unfitted meshes, e.g, a uniform Cartesian mesh, often requires technical adjustments to the basis function to enforce the jump condition with high accuracy [5, 17].

Note that the computational domain has been separated as the union of multiple non-overlapping subdomains, each of which corresponds to a local boundary value problem after endowing the interface with an appropriate boundary condition [52]. As a result, the non-overlapping Dirichlet-Neumann algorithm [52, 68, 69] is developed to address elliptic interface problems, where decomposed subproblems are alternatively solved using mesh-based numerical methods [3, 34, 35]. However, the complex geometry of subdomain interfaces remains a major concern during the meshing process. Fortunately, many domain decomposition methods [9, 52, 60] can be formulated at the continuous level, thereby making it computationally feasible to adopt the meshless deep learning technique [14, 29, 70] as the local problem solver [19]. Thanks to the rapid development of artificial intelligence science, much attention has recently been paid to combining deep learning with insights from domain decomposition methods. The physics-informed neural networks (abbreviated as PINNs in what follows) [29–31, 53], among others [56,70,72] has been utilized to solve Dirichlet and Neumann subproblems within the classical Dirichlet-Neumann algorithm [39], which is named "DeepDDM" and applied to several interface problems as a proof of concept. To further enhance its scalability properties, the DeepDDM method is extended with the aid of coarse space correction [45, 52]. Note that in the degenerate case of homogeneous jump conditions, the continuity of averaged solution between neighbouring subdomains, as well as its first and higher-order derivatives, are explicitly enforced through additional penalty terms in a series of papers [22,24,27,43,55,67]. Designing specific network architectures is another way of dealing with the complex geometry and jump condition [10, 21, 61, 64], e.g., using adaptive activation functions [25, 26], augmenting an additional coordinate variable as the input of the solution ansatz [32], replacing neural network structures with extreme learning machines [10, 11] or graph neural networks [59], to name a few. Additionally, an efficient hybrid approach [4, 64] is developed to address the singular and regular solutions using neural network and finite difference methods, respectively.

However, when solving the Dirichlet subproblem using neural networks, the gradient of trained model often exhibits higher errors at the boundary compared to its interior domain, which poses challenges when explicitly enforcing the flux transmission condition along subdomain interfaces. In contrast, a novel Dirichlet-Neumann learning algorithm using neural network extension operators is studied in this work for tackling high-contrast coefficients as well as irregular interfaces, alleviating the issue of inaccurate