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A Novel Multipoint Stress Control Volume Method for Linear Elasticity on Quadrilateral Grids

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Abstract. In this paper, we develop a novel control volume method that is locally conservative and locking-free for linear elasticity problem on quadrilateral grids. The symmetry of stress is weakly imposed through the introduction of a Lagrange multiplier. As such, the method involves three unknowns: stress, displacement and rotation. To ensure the well-posedness of the scheme, a pair of carefully defined finite element spaces is used for the stress, displacement and rotation such that the inf-sup condition holds. An appealing feature of the method is that piecewise constant functions are used for the approximations of stress, displacement and rotation, which greatly simplifies the implementation. In particular, the stress space is defined delicately such that the stress bilinear form is localized around each vertex, which allows for the local elimination of the stress, resulting in a cell-centered system. By choosing different definitions of the space for rotation, we develop two variants of the method. In particular, the first method uses a constant function for rotation over the interaction region, which allows for further elimination and results in a cell-centered system involving displacement only. A rigorous error analysis is performed for the proposed scheme. We show the optimal convergence for L^2-error of the stress and rotation. Moreover, we can also prove the superconvergence for L^2-error of displacement. Extensive numerical simulations indicate that our method is efficient and accurate, and can handle problems with discontinuous coefficients.

AMS subject classifications: 65N15, 65N12, 65N08, 65N30
Key words: Mixed finite element method, linear elasticity, stress elimination, local conservation.

1 Introduction

Mixed finite element methods in stress-displacement formulation for linear elasticity is popular in solid mechanics since they avoid locking and provide a direct approximation to the stress that is the primary physical interest. Numerous methods have been

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developed in the context of strong stress symmetry [9, 10, 18, 25] and weak stress symmetry [7, 8, 11, 29]. From the computational point of view, the introduction of the stress element will lead to a saddle-point system and result in additional computational costs, therefore, many of these methods could suffer from extra computational costs. The common approaches to overcome this issue lie in hybridization and reduction to cell-centered system. The former has been successfully applied in the context of nonconforming mixed finite element (MFE) methods [6] and hybridizable discontinuous Galerkin methods [26]. The latter is to design suitable strategies to treat the stress part such that the local elimination can be applied to obtain a reduced system. In this context, we mention in particular the multipoint stress mixed finite element method [4, 5]. Therein, MFE spaces with the lowest order Brezzi-Douglas-Marini (BDM_1) degrees of freedom is used for the stress and piecewise constant approximation is used for the displacement. In addition, the vertex quadrature rule is used for the computation of the stress bilinear form, which localizes the stress degrees of freedom. As such, the mass matrix for stress is block-diagonal, and thus allows local elimination of the stress. The resulting system is symmetric and positive definite, which enhances the computational efficiency. This method is motivated by the multipoint flux mixed finite element (MFMFE) method [21,27,28] for Darcy flow that is closely related to the multipoint flux approximation (MPFA) method [1–3, 14, 15]. The MFMFE method invokes BDM_1 on simplicial and quadrilateral grids. As an alternative, a MFEM based on broken Raviart-Thomas velocity space is proposed in [23,24]. All these methods share the similar idea that a vertex quadrature rule is applied for the computation of the mass matrix, which results in a block-diagonal mass matrix. Therefore, the flux can be locally eliminated, which leads to a cell-centered pressure system, rendering the method computationally attractive.

In this paper we aim to develop a new method that only uses piecewise constant approximations for the involved unknowns, that can be further reduced to a symmetric positive definite system. Our method is closely related to the multipoint stress mixed finite element method proposed in [4, 5], but with much simpler construction and implementation in the sense that no special quadrature rule is needed. The devising of the stress space is more subtle as it needs to be carefully balanced with the displacement such that the inf-sup condition holds. To this end, we divide the quadrilateral element into four smaller quadrilaterals by connecting the interior points to the midpoint of each edge and then define the stress space as a constant function over each smaller quadrilateral, and at the same time it is normal continuous over the edges lying on the quadrilaterals, but no continuity is imposed for the new edges generated by the subdivision. As a consequence, the bilinear form associated with the stress is localized around each vertex, which resembles the vertex quadrature rule used in multipoint stress mixed finite element method. The major difference is that we use piecewise constant function to avoid the quadrature rule. Then we develop two variants of the method by choosing different spaces for the rotation. In the first method, we let the rotation be a constant function over each interaction region formed by the four smaller quadrilaterals sharing the original vertex. Then the system can be further reduced to achieve a symmetric and positive definite cell-centered