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A 2D Fourth-Order CESE Scheme for Solving the Ideal MHD Equations

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Abstract. We construct a two-dimensional fourth-order space-time conservation element and solution element (CESE) schemes for solving the ideal magnetohydrodynamics (MHD) equations. In the CESE scheme, the flow variables are calculated by using the same procedure as that of the original second-order CESE scheme. The scheme preserves most favorable attributes of the original second-order CESE method. Moreover, it is simple and easy to program. The numerical example for the smooth Alfv*é*n wave problem suggests that the scheme can achieve the fourth-order accuracy for smooth solutions. In order to verify the efficiency of the schemes, we simulate several 2D MHD problems. We find that the fourth-order scheme can capture shocks and details of complex flow structures very well, and control the magnetic divergence efficiently. Moreover, the scheme is essentially CFL number insensitive schemes. The last several complex test problems further verify the performance of proposed scheme.

AMS subject classifications: 65M08, 76W05

Key words: CESE method, fourth-order accuracy, MHD.

1 Introduction

Magnetohydrodynamics (MHD) play an important role in many fields including astrophysics, space physics and plasma physics, etc. Since the solutions of the compressible MHD equations are characterized by complicated nonlinear wave structure and admit strong shocks and contact discontinuities, it is very difficult to analytically treat MHD equations. A primary approach to explore the physical mechanisms in MHD is numerical

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simulation. In the past few decades, the numerical study of MHD has attracted much attention, and various numerical methods have been developed for MHD equations, such as finite-difference method (FDM), finite volume method (FVM), and spectral methods, etc. [1,12–14,16,17,22]. The space-time conservation element and solution element (CESE) method [5,6,15,36] is a special finite-volume-type method for solving equations of conservation laws. However, the CESE method differs from other traditional well-established methods (FVM and FDM). Particularly for the high-order scheme, the CESE scheme avoids the common shortcomings of traditional high order schemes and demonstrates many favorable attributes, including 1) a unified treatment of space and time, achieving the same accuracy in time and space with a fully discrete one-stage formulation, 2) the use of a highly compact node stencil regardless of the order of accuracy, involving only the immediate neighboring cells surrounding the cell, and 3) the flux conservation in space and time without using an approximated Riemann solver. Moreover, the discontinuous Galerkin (DG) method with space-time evolution and Riemann-solver-free method also has these attractive features, such as STDG CVS (space-time discontinuous Galerkin Cell Vertex Schemes) [31]. Tu [31] compared three Riemann-solver-free Cell-Vertex schemes, including include the second-order space-time CESE CVS, high-order DG CVS and high-order semi-discrete Runge-Kutta DG CVS, for conservation laws. The interested reader can see it.

The CESE method was originally proposed by Chang and co-workers [5, 6]. Later, many extension and improvement of CESE schemes have been proposed [7,27,32,34–36]. Efforts have also been made to design higher-order CESE schemes [2, 8, 9, 21]. For example, Liu and Wang [21] developed an arbitrary-order one-dimensional CESE scheme based on arbitrary Taylor expansions in the solution elements. Chang [10] proposed a novel approach for constructing a highly-stable high-order CESE scheme. Bilyeu *et al.* [3] extended the original CESE method to the Euler solver for 2-D unstructured meshes in two-dimensions. In the methods of Chang [10] and Bilyeu *et al.* [3], the even-order derivatives were calculated by integrating the conservation law in the CEs and the odd-order derivatives were treated using a central difference scheme. Shen *et al.* [26] proposed high-order versions including third and fourth order for the Euler equation on hybrid grids in two-dimensions.The second- and third-order derivatives are calculated by a modified finite-difference/weighted-average procedure. They updated the high-order derivatives in a descending sequence from the highest order to the second order. Yang *et al.* [33] extended this CESE MHD solver to a fourth-order version based on [26]. However, the fourth-order version is constructed on uniform rectangle meshes in Cartesian coordinates. All the boundaries of the CEs are parallel to the coordinate surfaces, and the normal direction is along the coordinate axis. We extended the original CESE method to third order for 2D MHD equations in [39]. Moreover, this method can be directly applied to the unstructured meshes. In this paper, we extended the original CESE method to four order for 2D MHD equations. It can also be directly applied to the unstructured meshes.

In the CESE scheme, the variables and their spatial derivatives are treated as independent variables and are updated simultaneously by individual time marching schemes. In