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Energy Stable Gradient Flow Schemes for Shape and Topology Optimization in Navier-Stokes Flows

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Abstract. We study topology optimization governed by the incompressible Navier-Stokes equations using a phase field model. Unconditional energy stability is shown for the gradient flow in continuous space. The novel generalized stabilized semi-implicit schemes for the gradient flow in first-order time discretization of Allen-Cahn and Cahn-Hilliard types are proposed to solve the resulting optimal control problem. With the Lipschitz continuity for state and adjoint variables, the energy stability for time and full discretization has been proved rigorously on the condition that the stabilized parameters are larger than specific values. The proposed gradient flow scheme can work with large time steps and exhibits a constant coefficient system in full discretization, which can be solved efficiently. Numerical examples in 2d and 3d show the effectiveness and robustness of the optimization algorithms proposed.

AMS subject classifications: 76D55, 49M05, 65M12, 76D05

Key words: Topology optimization, incompressible Navier-Stokes equations, stabilized gradient flow, energy stability, phase field method.

1 Introduction

Shape and topology optimization [3] of computational fluid dynamics is a popular research topic with applications such as auto-coronary bypass anastomoses in medical science [37], laminar flow wing design in aeronautics [33], and pipe flow [11]. Such optimal control problems in fluid flow [33] aim to seek a configuration or layout for optimizing some objective (e.g., energy dissipation and geometric inverse problem) subject to geometric and physical constraints such as incompressible fluid flows [12,15,22,30,31,34,39]

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or compressible Navier-Stokes equations [35]. Compared to shape optimization by adjusting the profile of geometric boundary to obtain a better configuration [24], topology optimization can perform both shape and topological changes of structures. Numerical simulation of topology optimization in fluid flows can be realized via variable density method [5], topological derivative [25, 38], level set method [4, 14, 49, 50], phase field method [19,29], etc.

As a diffusive interface tracking technique, the phase field method for structural topology optimization (see, e.g., [27, 28, 43, 45, 46]) has been introduced for minimizing general volume and surface functionals constrained by incompressible Navier-Stokes flows [2, 19–21]. The porous medium approach [5] proposed by Borrvall and Petersson enabled the governing equations to be defined on a fixed domain with a variable term characterizing the inverse permeability at each point. The main idea of topology optimization in the phase field model is to combine the objective and the so-called Ginzburg–Landau energy to construct the total free energy. The latter can be regarded as a diffuse interface approximation of perimeter regularization, which implies the existence of solutions to the resulting optimal control problem [20]. For topology optimization constrained by Navier-Stokes flows, Garcke et al. [19,21] discussed the differentiability of the solution with respect to (w.r.t.) the phase field function, the first-order necessary condition from sensitivity analysis, and sharp interfacial asymptotic analysis. These results provide reliable support for the subsequent analysis of our proposed optimization algorithm.

The gradient flow method of the phase field model is a powerful tool for the minimization of nonlinear or multi-physical coupling types of energy functionals for many problems (see, e.g., [13, 40, 44]). From the numerical perspective, a gradient flow scheme is generally evaluated on the energy dissipation and its computational efficiency [40]. The energy dissipative gradient flow scheme is of crucial importance to topology optimization in the sense that the sequence generated by the algorithm is convergent and monotonous, which reduces oscillations of the objective during shape evolution. Different from phase field models in e.g., physical interface dynamics [2] or crystallization [16], the difficulty of constructing an energy dissipative gradient flow scheme for topology optimization arises from the coupling among the gradient flow, linear/nonlinear physical constraints of partial differential equations and possible extra adjoint systems. For shape design constrained by Stokes flows in the phase field model [29, 32], energy monotonic-decaying gradient flow schemes were proposed for minimization of energy dissipation via the stabilized method (see, e.g., [47,48]). An efficient iterative thresholding method [9] was developed for topology optimization in Stokes flows. To the best of our knowledge, there is no research on energy dissipative gradient flow schemes for topology optimization constrained by the nonlinear partial differential equations, such as the incompressible Navier-Stokes flows.

In this paper, we derive the gradient flow of the phase field model for topology optimization of incompressible Navier-Stokes flows and show the energy dissipation property in continuous space by overcoming the difficulties caused by the introduction of the