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High-Order Accurate Entropy Stable Finite Difference Schemes for the Compressible Euler Equations with the van der Waals Equation of State on Adaptive Moving Meshes

Shangting Li[1] and Huazhong Tang[2,*]

[1] *CAEP Software Center for High Performance Numerical Simulation, Beijing 100088; Institute of Applied Physics and Computational Mathematics, Beijing 100088, P.R. China.*

[2] *Center for Applied Physics and Technology, HEDPS and LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China.*

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Abstract. This paper develops the high-order entropy stable finite difference schemes for multi-dimensional compressible Euler equations with the van der Waals equation of state on adaptive moving meshes. Semi-discrete schemes are first nontrivially constructed on the newly derived high-order entropy conservative (EC) fluxes in curvilinear coordinates and scaled eigenvector matrices as well as the multi-resolution WENO reconstruction, and then the fully-discrete schemes are given by using the high-order explicit strong-stability-preserving Runge-Kutta time discretizations. The high-order EC fluxes in curvilinear coordinates are derived by using the discrete geometric conservation laws and the linear combination of the two-point symmetric EC fluxes, while the two-point EC fluxes are delicately selected by using their sufficient condition, the thermodynamic entropy and the technically selected parameter vector. The adaptive moving meshes are iteratively generated by solving the mesh redistribution equations, in which the fundamental derivative related to the occurrence of non-classical waves is involved to produce high-quality mesh. Several numerical tests are conducted to validate the accuracy, the ability to capture the classical and non-classical waves, and the high efficiency of our schemes in comparison with their counterparts on the uniform mesh.

AMS subject classifications: 65M06, 35L02, 76M20

Key words: Entropy stable scheme, entropy conservative scheme, mesh redistribution, van der Waals equation of state.

*Corresponding author. *Email addresses:* `li_shangting_caep@163.com` (S.T. Li), `hztang@math.pku.edu.cn` (H.Z. Tang)

1 Introduction

This paper is concerned with high-order accurate entropy stable (ES) schemes for the compressible Euler equations with the van der Waals equation of state (EOS), which are given by

$$\frac{\partial \boldsymbol{U}}{\partial t}+\sum_{k=1}^{d}\frac{\partial \boldsymbol{F}_k(\boldsymbol{U})}{\partial x_k}=0,\quad d=1,2,3, \tag{1.1}$$

$$\boldsymbol{U}=(\rho,\rho\boldsymbol{v}^\top,E)^\top, \tag{1.2}$$

$$\boldsymbol{F}_k=\left(\rho v_k,\rho v_k\boldsymbol{v}^\top+p\boldsymbol{e}_k^\top,(E+p)v_k\right)^\top, \tag{1.3}$$

and

$$p=\frac{\rho RT}{1-\rho b}-a\rho^2,\quad e=c_vT-a\rho, \tag{1.4}$$

where ρ, $\boldsymbol{v}=(v_1,\cdots,v_d)^\top$, T, and $E=\rho e+\rho|\boldsymbol{v}|^2/2$ are the density, the velocity vector, the temperature, and the total energy, respectively. Moreover, $\boldsymbol{e}_k$ denotes the k-th column of the $d\times d$ unit matrix, e is the specific internal energy, R is the positive gas constant, c_v is the specific heat at constant volume, and $a\geqslant 0$ and $b\geqslant 0$ are two van der Waals constants depending on the intermolecular forces and the size of the molecules. Obviously, when $a=0$ and $b=0$, the van der Waals EOS (1.4) reduces to the ideal gas law. The van der Waals EOS is often used to depict the potentially non-classical phenomena (occurrences of non-classical waves) above the saturated vapour curve near the thermodynamic critical point and has important applications in engineering such as heavy gas wind tunnels and organic Rankine cycle engines [2,18,56]. Several researchers investigated the creation and evolution of the non-classical waves [4,19,20,36,46,61], and studied the numerical schemes such as the total variation diminishing MacCormack (TVDM) methods [3,9,10] and the Roe type schemes [1,17,27,29,30] for the compressible Euler equations with the van der Waals EOS.

The van der Waals EOS (1.4) can be rewritten into a cubic equation with respect to the specific volume $\nu:=1/\rho$ as follows:

$$\nu^3-\left(b+\frac{RT}{p}\right)\nu^2+\frac{a}{p}\nu-\frac{ab}{p}=0, \tag{1.5}$$

so that the roots for the specified gas at a given pressure are no more than three cases: three identical real roots, three different real roots, and one real root and two imaginary roots [44]. The thermodynamic critical point at the critical temperature T_c (where the derivatives p_ν and $p_{\nu\nu}$ are zero) corresponds to the case of that the cubic equation (1.5) has three identical real roots and thus the van der Waals constants a and b can be obtained by analyzing (1.5) at such critical point as follows:

$$b=\frac{1}{3\rho_c},\quad a=\frac{9p_c}{8Z_c\rho_c^2},\quad Z_c=\frac{p_c}{R\rho_cT_c}=\frac{3}{8}, \tag{1.6}$$