Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0173

Vol. **38**, No. 3, pp. 661-683
September 2025

A Network Based Approach for Unbalanced Optimal Transport on Surfaces

Jiangong Pan[1], Wei Wan[2], Yuejin Zhang[1], Chenlong Bao[3,4] and Zuoqiang Shi[3,4,*]

[1] *Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China.*
[2] *School of Mathematics and Physics, North China Electric Power University, Beijing, China.*
[3] *Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China.*
[4] *Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing, 101408, China.*

Received 31 July 2024; Accepted (in revised version) 2 April 2025

Abstract. In this paper, we present a neural network approach to address the dynamic unbalanced optimal transport problem on surfaces with point cloud representation. For surfaces with point cloud representation, traditional method is difficult to apply due to the difficulty of mesh generating. Neural network is easy to implement even for complicate geometry. Moreover, instead of solving the original dynamic formulation, we consider the Hamiltonian flow approach, i.e. Karush-Kuhn-Tucker system. Based on this approach, we can exploit mathematical structure of the optimal transport to construct the neural network and the loss function can be simplified. Extensive numerical experiments are conducted for surfaces with different geometry. We also test the method for point cloud with noise, which shows stability of this method. This method is also easy to generalize to diverse range of problems.

AMS subject classifications: 65K10, 68T05, 68T07

Key words: Unbalanced optimal transport, Hamiltonian flow, point cloud, neural network.

1 Introduction

The concept of optimal transport (OT) stands as a foundational cornerstone, offering profound insights across numerous domains, including economics, physics, image processing, and machine learning [1]. At its essence, OT is concerned with devising the most

*Corresponding author. *Email addresses:* `mathpjg@sina.com` (J. Pan), `weiwan@ncepu.edu.cn` (W. Wan), `zhangyj19@mails.tsinghua.edu.cn` (Y. Zhang), `clbao@tsinghua.edu.cn` (C. Bao), `zqshi@tsinghua.edu.cn` (Z. Shi)

efficient means of reallocating resources from source to target allocation points while minimizing associated costs. While the conventional perspective on the OT problem revolves around achieving equilibrium, wherein the total mass of the source distribution aligns precisely with the target distribution, real-world scenarios often disrupt this equilibrium, resulting in resource distributions that introduce disparities in mass. These real-world scenarios have given rise to the development of the unbalanced optimal transport (UOT) problem, an extension designed to address situations characterized by unequal source and target distribution masses [2].

In the unbalanced rendition of the OT problem, the central objective of improving efficiency and minimizing transportation costs remains unwavering. However, the introduction of varying masses introduces an additional layer of complexity, presenting both challenges and avenues for exploration. This multifaceted issue finds applications across a spectrum of fields, including image registration [3–5], transformation and generation [6–8], as well as climate modeling [9–11], style transfer [8,12,13], and medical imaging [14–16]. Despite achieving notable numerical successes, the UOT problem grapples with computational constraints, as diverse conditions often pose formidable challenges for conventional methods, which severely limits the applicability of UOT in different scenarios.

OT problem has three different formulation: Monge problem, Kantorovich problem and Benamou-Brenier problem. In 1781, Monge formulated OT problem as an optimization problem of minimizing the cost functional over all feasible transport plan [17]. However, solving the Monge problem directly has proven to be a formidable task, primarily due to its intricate non-convex nature and the absence of minimal solutions.

To address this challenge, Kantorovich introduce a new formulation by relaxing the transport plan to joint distribution [18]. With this elegant relaxation, OT problem can be formulated as a linear programming problem which can be solved efficiently. Many powerful algorithms have been developed based on Kantorovich formulation, such as Sinkhorn method [1] etc.

Benamou and Brenier [19] made a groundbreaking contribution by introducing the dynamic formulation into the optimal transport problem. Consider the model over a time interval $T=1$ and a spatial region Ω. Here, ρ represents the density, and $\boldsymbol{v}$ denotes the velocity of the density. In the dynamic OT problem, we are concerned with ρ, subject to given initial and terminal densities ρ_0 and ρ_1, and $(\rho,\boldsymbol{v})$ must satisfy the mass conservation law. The primary objective of OT is to minimize the total cost across all feasible pairs of $(\rho,\boldsymbol{v})\in\mathcal{C}(\rho_0,\rho_1)$. The problem can be formally stated as follows:

$$\mathcal{W}_2(\rho_0,\rho_1)=\min_{(\rho,\boldsymbol{v})\in\mathcal{C}(\rho_0,\rho_1)}\int_0^1\int_\Omega\frac{1}{2}\rho(t,\boldsymbol{x})\|\boldsymbol{v}(t,\boldsymbol{x})\|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t, \tag{1.1}$$

where

$$\mathcal{C}(\rho_0,\rho_1):=\{(\rho,\boldsymbol{v}):\partial_t\rho+\mathrm{div}(\rho\boldsymbol{v})=0,\ \rho(0,\boldsymbol{x})=\rho_0(\boldsymbol{x}),\ \rho(1,\boldsymbol{x})=\rho_1(\boldsymbol{x})\}. \tag{1.2}$$